Sequential Screening with Type-Enhancing Investment*

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Abstract

Due to the well-known efficiency–rent extraction trade-off, the optimal mechanism in a pure screening environment (e.g., revenue maximization in auctions or cost minimization in procurement) typically calls for distortions in allocative efficiency when agents possess private information at the time of contracting. In this paper, we introduce first-stage type-enhancing hidden investment to a standard sequential screening model of procurement, and find that (1) with convex investment cost, mitigation of allocative distortion must arise; and (2) such mitigation can even be extreme with linear investment cost—procurement cost minimization must require social efficiency when the investment is sufficiently effective.

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1 Introduction

A central insight in the screening literature is that when agents possess private information at the time of contracting, the optimal allocation (e.g., revenue maximization in auctions or cost minimization in procurement) would require distortion from the efficient level (only no distortion at the "top"), due to the long-recognized efficiency–rent extraction trade-off. In a typical environment

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of sequential screening—e.g., the two-stage settings of Courty and Li [3] and Eső and Szentes [4]—agents are endowed at the time of contracting with a private type (the first-stage type), which determines the distribution of their second-stage type. As such, the central insight implies that at the optimum, the second-stage allocation is in general discriminatory over the first-stage type to optimally balance information elicitation and rent extraction in a dynamic setting.

This paper shows that when introducing first-stage type-enhancing hidden investment to a sequential screening model of procurement, the mitigation of allocative distortion necessarily arises. We further show that such mitigation can even be extreme: With linear investment cost, the procurement cost-minimizing mechanism, surprisingly, could entail social welfare maximization. As a result, there is neither a loss of allocative efficiency nor distortion in investment (compared with the first-best level). One practical motivation for such type-enhancing investment comes from the observation that situations are abundant in which agents can conduct hidden actions in the first stage to gain a better distribution of the second-stage type. For example, in a procurement setting, the supplier can conduct R&D to improve his chance of discovering a more cost-efficient way to deliver the product procured. The level of R&D investment of the agent (the supplier) is usually not observed by the principal (the procurer).

We demonstrate our result by considering a two-stage procurement model with a procurer (she, the principal) and a supplier (he, the agent). The procurer wishes to procure a product from the supplier, which she can acquire alternatively from an outside option at cost c_0. In the first stage, the agent does not know exactly his cost of producing the product. He is instead endowed with a capacity (i.e., his first-stage private type) that determines the distribution of his production cost in the second stage (i.e., his second-stage type). The agent's first-stage type is his private information, which can be determined by, for example, the agent's lab capacity, quality of facilities, or technology endowment. A higher type means a better distribution of production costs in the sense of first-order stochastic dominance. The agent can make an optimal level of unobservable investment to improve his first-stage type (i.e., increase his capacity) so that his second-stage production cost will be drawn from an improved ex post distribution. The contract is offered at the first stage, and the type-enhancing investment is conducted by the agent after the contract is accepted. At the second stage, the production cost is drawn from an ex post distribution determined by the improved type; again, the realized production cost is the agent's private information. The principal's objective is to design a contract to minimize her expected procurement cost.

We show that the mitigation of allocative distortion is a general phenomenon by investigating nonlinear investment cost functions. The main driving force for such mitigation is the following investment-enhancing effect: Not excluding sufficiently efficient types from trade boosts the agent's investment incentive, which benefits the principal. With linear investment cost, this investment-enhancing effect can be so dominant over the information-rent-saving consideration that the principal should never exclude any efficient second-stage type from trade. In other words, the mitigation can be pushed to the extreme such that ex post efficiency can be restored.

Specifically, with linear investment cost, when the marginal investment cost is relatively small, the procurement cost-minimizing mechanism is exactly the efficient mechanism. As such, the optimal mechanism is a static one that only relies on the second-stage type—the benefit of sequential screening is completely absent, so the principal loses the power of screening at the first stage. More importantly, the optimal mechanism is ex post efficient. The usual efficiency–rent extraction trade-off is *absent* in this environment. More precisely, we find that when all first-stage types have incentive to make investment, each first-stage type's information rent is *independent* of the allocation rule,[1] and moving the allocation rule toward the efficient level will not alter any first-stage type's information rent. Thus, the investment-enhancing effect always dominates the information-rent-saving consideration such that the procurement cost-minimizing mechanism must be efficient.

Our paper contributes to the literature on dynamic mechanism design, which originates in the seminal work of Baron and Besanko [1]. Courty and Li [3] and Eső and Szentes [4] demonstrate, in different environments on sequential screening with pure adverse selection, that at the optimum, the second-stage mechanism is discriminatory across first-stage types. Krähmer and Strausz [6] introduce endogenous information acquisition to the monopolistic price discrimination model of Courty and Li [3]. Pavan, Segal, and Toikka [12] provide a general treatment of optimal dynamic mechanism design with pure adverse selection. Li and Shi [8] and Guo, Li, and Shi [5] study discriminatory information disclosure in sequential screening with pure adverse selection. Our paper contributes to this strand of the literature by studying a mixed adverse selection and moral hazard problem in a dynamic environment, in which the agent can make investment in the first stage to change the distribution of his second-stage type.

Our paper essentially introduces moral hazard in the first stage to the seminal work of Courty and Li [3] on sequential screening. Courty and Li [3] study monopolistic pricing discrimination,

[1] Details are provided in Section 4.5.

in which consumers only know the distribution of their valuations at the time of contracting and learn the actual valuations in the second stage. We adopt a parallel procurement setting, assuming that at the time of contracting, the agent only knows the distribution of his production cost and subsequently discovers his actual cost. However, our setting departs from that of Courty and Li [3] by further taking into account the possibility that the agent can also make some unobservable investment to improve his first-stage type. One insight in Courty and Li [3] is that at the optimum, the second-stage mechanism discriminates over first-stage types. Such discrimination yields efficiency loss in the second stage. We show that ex post efficiency can be improved by introducing first-stage type-enhancing hidden action; moreover, the efficiency can even be completely restored when the investment cost function is linear. This ex post efficiency result differs from the distortive allocation result in Liu and Lu [9], who also introduce moral hazard to a sequential screening model. The key difference is that in Liu and Lu [9], the first-stage type is the marginal cost of exerting effort, and the effort fully determines the distribution of the second-stage type. Therefore, the agent's information rent depends on the second-stage allocation rules, so the usual efficiency–rent extraction trade-off is still present there. However, here the key driving force for our result is the absence of such a trade-off for the case of linear investment cost function.

In conventional sequential screening settings but with ex post participation constraints, Krähmer and Strausz [7] and Bergemann, Castro, and Weintraub [2] demonstrate that under some conditions, the optimal contract is a static one that only screens the agent's second-stage type. Our result suggests another possibility for such independence of the first-stage type at the optimum: When allowing the possibility of a type-enhancing hidden action, the optimal contract can also only respond to the agent's second-stage type.

The rest of the paper is organized as follows. Section 2 sets up the model. We study the first-best benchmark in Section 3. Section 4 presents the main analysis, and Section 5 concludes. The appendix collects some technical proofs.

2 The Model

A risk-neutral buyer (the principal, she) wishes to procure a product from a risk-neutral supplier (the agent, he). At the time of contracting, the agent does not know his exact cost of supplying the product. However, the agent has some private signal, θ, that determines his future production efficiency. Specifically, production cost c is randomly drawn from a cumulative distribution function

$H(\cdot,\theta)$ (parameterized by θ) whose support is $[\underline{c},\overline{c}]$ with $0 \leq \underline{c} < \overline{c} \leq \infty$. Roughly speaking, higher θ means that the agent would have a higher expected efficiency in delivering the product. The exact meaning will become clear soon. From the principal's perspective, θ is randomly drawn from a CDF $G(\cdot)$ with density function $g(\cdot) > 0$ everywhere over the support $[\underline{\theta},\overline{\theta}]$, where $\underline{\theta} \geq 0$.

Departing from the classical sequential screening setting, here the agent can make unobservable investment $\alpha \geq 0$ at the cost of $C(\alpha)$ to increase his type from θ to $\theta+\alpha$, where $C(\cdot)$ is a twice continuously differentiable function over $[0,+\infty)$, with $C(0) = 0$, $C'(\cdot) > 0$ and $C''(\cdot) \geq 0$ in $[0,+\infty)$. That is, with investment α, production cost c will be drawn from distribution $H(\cdot,\theta+\alpha)$. The cost of producing the product, which is privately observed by the agent, is realized after the investment. The agent's delivery cost c is incurred only when the principal acquires the product from him. When the trade does not occur between the principal and the agent, the principal exercises her outside option for the product at cost c_0 with $c_0 \in (\underline{c},\overline{c})$. $C(\cdot)$, $G(\cdot)$, $H(\cdot,\cdot)$, and c_0 are public information. The principal's objective is to minimize her expected procurement cost.

The timing of the game is as follows.

Time 0: The agent is privately informed about his type θ.

Time 1: The principal offers a contract and she commits to it. If the agent rejects, then the game ends and he obtains his reservation utility, which is normalized as zero; the principal exercises her outside option for the product at cost c_0. If the agent accepts, he decides how much investment α to make to improve his first-stage type.

Time 2: The agent's private delivery cost is drawn from $H(\cdot,\theta+\alpha)$. The contract is then executed.

Following the revelation principle (Myerson [11]), we restrict to direct mechanisms $\{p(\theta,c), y(\theta,c), \alpha(\theta)\}$, in which θ and c are the agent's first-stage and second-stage reports, respectively; here $p(\theta,c)$ is the acquisition probability, $y(\theta,c)$ is the payment to the agent, and $\alpha(\theta)$ is the principal's recommendation of effort to the agent.[2] More details can be found in Section 4.

We assume that $H(c,z)$ is twice continuously differentiable in $(c,z) \in [\underline{c},\overline{c}] \times [\underline{\theta},+\infty)$ and that the density function $h(c,z)$ (i.e., $H_1(c,z)$) is strictly positive everywhere over the support.[3] In addition,

$$H_2(c,z) > 0, H_{22}(c,z) < 0, \forall c \in (\underline{c},\overline{c}).$$

[2] We use "effort" and "investment" interchangeably in this paper.

[3] For a two-variable function, we use subscripts 1 and 2 to represent the partial derivative with respect to the first and second argument, respectively.

Positive $H_2(c,z)$ means that higher z leads to a better cost distribution in the sense of first-order stochastic dominance; this is exactly what we mean by saying "a higher θ means a better expected production efficiency" at the beginning of this section. Negative $H_{22}(c,z)$ means that the marginal effect of z decreases.[4] Our formulation of $H(c,z)$ covers the following widely adopted CDF as a special case:

$$H(c,z) = 1 - (1 - F(c))^{z+\beta_0}, \tag{1}$$

where $F(c)$ is a CDF with strictly positive density function everywhere over the support $[\underline{c}, \overline{c}]$, and $\beta_0 > 0$ is a constant.

We make the following regularity assumption, as in the dynamic mechanism design literature.

Assumption 1 (Regularity)**.** *The virtual cost function* $J(c,\theta) = c + \frac{1-G(\theta)}{g(\theta)} \frac{H_2(c,\theta)}{h(c,\theta)}$ *is strictly increasing in* c *when* $c \in [\underline{c}, c_0]$ *and strictly decreasing in* θ*.*

Assumption 1 is satisfied when, for example, (i) the hazard rate $\frac{g(\theta)}{1-G(\theta)}$ is strictly increasing in θ; (ii) $H_2(c,\theta)/h(c,\theta)$ is increasing in c; and (iii) $H_2(c,\theta)/h(c,\theta)$ is decreasing in θ. The monotone hazard rate assumption is standard in the mechanism design literature, and the latter two assumptions parallel Assumption 1 and Assumption 2, respectively, in Eső and Szentes [4].

Note that when no investment is allowed—i.e., the agent's first-stage type is θ and his second-stage type c is a random draw from $H(\cdot,\theta)$—our setting reduces to a standard sequential screening setting, as in Courty and Li [3] and Eső and Szentes [4]. Given Assumption 1, standard arguments, as in these two papers,[5] imply that the cost-minimizing mechanism is deterministic and has the feature that for each θ, the trade happens if and only if the virtual cost $J(c,\theta) \le c_0$. In other words, the trade happens if and only if the realized second-stage type $c \le c_B(\theta)$, where $c_B(\theta) \in (\underline{c}, \overline{c})$ is uniquely characterized by

$$J(c_B(\theta), \theta) - c_0 = 0. \tag{2}$$

It is easy to see that $c_B(\theta)$ is strictly increasing in θ and that $c_B(\theta) \le c_0$ with equality only when $\theta = \overline{\theta}$. Therefore, when no investment is allowed, the optimal allocation distorts from the efficient level c_0, leading to efficiency loss.

[4] This assumption validates the "first-order approach," which replaces the agent's moral hazard incentive compatibility with the first-order condition.

[5] The proof is available from the authors upon request.

3 The First-best Benchmark

We first study the first-best benchmark, in which the agent's types and his effort level are public information. Suppose that for the agent with type θ and realized cost c, the contract specifies the payment to the agent $y^{FB}(\theta, c)$ and the acquisition probability $p^{FB}(\theta, c)$. In the first stage, the contract prescribes the agent's investment level $\alpha^{FB}(\theta)$.

The social cost for type θ is

$$C(\alpha^{FB}(\theta)) + \int_{\underline{c}}^{\overline{c}} \left[p^{FB}(\theta, c)c + (1 - p^{FB}(\theta, c))c_0\right] h(c, \theta + \alpha^{FB}(\theta))dc. \tag{3}$$

It is obvious that the first-best allocation is ex post efficient:

$$p^{FB}(\theta, c) = \begin{cases} 1, \text{ when } c \leq c_0 \\ 0, \text{ when } c > c_0 \end{cases}.$$

Thus, the social cost further boils down to

$$C(\alpha^{FB}(\theta)) - \int_{\underline{c}}^{c_0} H(c, \theta + \alpha^{FB}(\theta))dc + c_0, \tag{4}$$

which is strictly convex in $\alpha^{FB}(\theta)$. Therefore, $\alpha^{FB}(\theta)$ is unique and satisfies

$$C'(\alpha^{FB}(\theta)) - \int_{\underline{c}}^{c_0} H_2(c, \theta + \alpha^{FB}(\theta))dc \geq 0, \text{ with equality when } \alpha^{FB}(\theta) > 0. \tag{5}$$

The following proposition summarizes.

Proposition 1. *The first-best allocation is ex post efficient. The first-best investment of type θ, $\alpha^{FB}(\theta)$, is characterized by (5).*

4 Analysis of the Cost-minimizing Mechanism

Now we turn to the original problem. There is no loss of generality to focus on truthful direct mechanisms, according to Myerson [11]. In the first stage, when the agent reports $\hat{\theta}$, he receives an investment recommendation $\alpha(\hat{\theta}) \geq 0$. The agent decides on investment level α after reporting $\hat{\theta}$. In the second stage, his delivery cost c is realized according to $H(\cdot, \theta + \alpha)$, where θ is his true first-stage type; he further reports his cost realization. The report is denoted by $\hat{c}$. Then the payment rule $y(\hat{\theta}, \hat{c})$ and the acquisition probability $p(\hat{\theta}, \hat{c})$ are executed.

4.1 Stage Two

Assuming truthfully reported θ in stage one, suppose that the agent's true provision cost is c, but he reports $\hat{c}$. Let $\tilde{\pi}(\theta, \hat{c}, c)$ be his expected payoff in stage two. Then

$$\tilde{\pi}(\theta, \hat{c}, c) = y(\theta, \hat{c}) - p(\theta, \hat{c})c. \tag{6}$$

Envelope theorem yields

$$\frac{d\tilde{\pi}(\theta, c, c)}{dc} = \frac{\partial \tilde{\pi}(\theta, \hat{c}, c)}{\partial c}|_{\hat{c}=c} = -p(\theta, c),$$

which leads to

$$\tilde{\pi}(\theta, c, c) = \tilde{\pi}(\theta, \overline{c}, \overline{c}) + \int_{c}^{\overline{c}} p(\theta, s) ds, \ \forall c, \forall \theta. \tag{7}$$

It is clear that the second-stage incentive compatibility (IC) is equivalent to that (7) holds and that $p(\theta, c)$ is decreasing in c for any fixed θ. Note that if the agent misreported his type in stage one as $\hat{\theta}$, he will still truthfully report c in stage two, given that the IC holds in the second stage upon the first stage being truthful. This is because the agent's first-stage type does not directly enter his second-stage payoff.

4.2 Stage One

The first-stage IC requires that the agent report his type truthfully and follow the principal's recommendation on investment level. Note that when the agent reports his type and then receives the recommendation (which depends on the report), he always chooses a unique optimal effort level regardless of the recommendation he receives. This is because the agent's belief is not affected by the recommendation and, as we will show in the derivation of the moral hazard constraint (10) (in Appendix A), the agent's payoff is (strictly) concave in effort, so that he will not randomize his effort level. Such an effort level only depends on his true type and the type he reported to the principal.

If the agent with type θ reports $\hat{\theta}$ and invests α, his expected payoff is

$$\hat{\pi}(\alpha, \hat{\theta}, \theta) = -C(\alpha) + \int_{\underline{c}}^{\overline{c}} \tilde{\pi}(\hat{\theta}, c, c) h(c, \theta + \alpha) dc.$$

In the derivation of the moral hazard constraint (10) (in Appendix A), we will show that $\hat{\pi}(\alpha, \hat{\theta}, \theta)$ is (strictly) concave in α. Let $\alpha(\hat{\theta}, \theta) = \arg\max_{\alpha \geq 0} \hat{\pi}(\alpha, \hat{\theta}, \theta)$,[6] and $\pi(\hat{\theta}, \theta) = \max_{\alpha \geq 0} \hat{\pi}(\alpha, \hat{\theta}, \theta) = \hat{\pi}(\alpha(\hat{\theta}, \theta), \hat{\theta}, \theta)$, which is the agent's expected utility when his true type is θ but he reports $\hat{\theta}$, given that he will respond optimally by taking $\alpha(\hat{\theta}, \theta)$ when receiving the recommendation $\alpha(\hat{\theta})$. The first-stage IC then requires that

$$\pi(\theta, \theta) \geq \pi(\hat{\theta}, \theta), \forall \theta, \hat{\theta}, \tag{8}$$

which we call the IC_1 constraint. Note that $\alpha(\hat{\theta}, \theta)$ is determined as follows:[7]

$$-C'(\alpha(\hat{\theta}, \theta)) + \int_{\underline{c}}^{\overline{c}} p(\hat{\theta}, c) H_2(c, \theta + \alpha(\hat{\theta}, \theta)) dc \leq 0, \text{ with equality if } \alpha(\hat{\theta}, \theta) > 0, \forall \theta, \hat{\theta}. \tag{9}$$

When the type θ agent reports truthfully, the principal's recommendation $\alpha(\theta)$ must coincide with the agent's optimal effort choice (obedience); i.e., $\alpha(\theta) = \alpha(\theta, \theta)$. This is characterized by the following first-order condition:

$$-C'(\alpha(\theta)) + \int_{\underline{c}}^{\overline{c}} p(\theta, c) H_2(c, \theta + \alpha(\theta)) dc \leq 0, \text{ with equality if } \alpha(\theta) > 0, \forall \theta, \tag{10}$$

which we call the moral hazard constraint MHC. IC_1 and MHC constitute the first-stage IC constraints.

Given (9), we obtain the following necessary conditions for the first-stage IC (all proofs are relegated to Appendix A).

Lemma 1. *The first-stage IC implies*

(i) Envelope condition:

$$\pi(\theta, \theta) = \pi(\underline{\theta}, \underline{\theta}) + \int_{\underline{\theta}}^{\theta} \int_{\underline{c}}^{\overline{c}} p(s, c) H_2(c, s + \alpha(s)) dc ds, \ \forall \theta.$$

(ii) The recommendation $\alpha(\theta)$ satisfies the moral hazard constraint MHC (10).

[6] $\alpha(\hat{\theta}, \theta)$ exists because $\lim_{\alpha \to +\infty} \hat{\pi}(\alpha, \hat{\theta}, \theta) = -\infty$.

[7] Details can be found in (19).

4.3 The Principal's Problem

By Lemma 1, the expected cost of procurement, which is the sum of the expected social cost and the agent's expected utility, can be written as

$$TC = \pi(\underline{\theta}, \underline{\theta}) + \int_{\underline{\theta}}^{\overline{\theta}} \left\{ \begin{array}{c} C(\alpha(\theta)) + c_0 \\ + \int_{\underline{c}}^{\overline{c}} p(\theta, c) \left[c + \frac{1-G(\theta)}{g(\theta)} \frac{H_2(c, \theta+\alpha(\theta))}{h(c, \theta+\alpha(\theta))} - c_0 \right] h(c, \theta + \alpha(\theta)) dc \end{array} \right\} g(\theta) d\theta. \quad (11)$$

The term in the square bracket,

$$c + \frac{1 - G(\theta)}{g(\theta)} \frac{H_2(c, \theta + \alpha(\theta))}{h(c, \theta + \alpha(\theta))},$$

is the adjusted virtual cost, denoted as $\tilde{J}(c, \theta, \theta + \alpha(\theta))$, with $\theta + \alpha(\theta)$ as the "effective first-stage type," which resembles the standard virtual cost in the classical sequential screening model.

The principal's problem can be expressed as

$$\min_{\{\alpha(\theta) \geq 0,\ p(\theta, c) \in [0,1],\ y(\theta, c)\}} TC$$

subject to

$$\text{constraints } IC_1 \text{ of (8) and } MHC \text{ of (10);} \quad (12)$$

$$\pi(\theta, \theta) \geq 0, \forall \theta; \quad (13)$$

$$\tilde{\pi}(\theta, c, c) = \tilde{\pi}(\theta, \overline{c}, \overline{c}) + \int_{c}^{\overline{c}} p(\theta, s) ds, \forall c, \forall \theta; \quad (14)$$

$$p(\theta, c) \text{ is decreasing in } c \text{ for any } \theta. \quad (15)$$

(12) contains the first-stage IC constraints; (13) is the first-stage IR constraint, and (14) and (15) are the equivalent conditions for the second-stage IC constraint. We call this Problem (O).

4.4 Mitigation of Distortion

In this section, we establish the general phenomenon that with first-stage type-enhancing investment, the second-stage allocative distortion will be mitigated for all first-stage types—the optimal second-stage allocative cutoff $c^*(\theta)$ will be (weakly) higher than the cutoff $c_B(\theta)$ without investment. As our goal is to investigate how the introduction of type-enhancing investment affects the distortion, we focus on deterministic second-stage mechanisms. In other words, $p(\theta, c) \in \{0,1\}$, $\forall\theta, \forall c$. Obviously, the second-stage IC then implies that for each first-stage type θ, there is a second-stage allocative cutoff $c(\theta) \in [\underline{c}, \overline{c}]$ such that $p(\theta, c) = 1$ when $c \le c(\theta)$; $p(\theta, c) = 0$ when $c > c(\theta)$. As such, we denote a deterministic mechanism as $\{c(\theta), y(\theta, c), \alpha(\theta)\}$ and call it a *cutoff mechanism.*

4.4.1 An Equivalent Characterization of Incentive Compatibility

The following result, which provides a *necessary and sufficient* condition for a cutoff mechanism being IC, is important for establishing the mitigation of distortion result.

Lemma 2. *A cutoff mechanism $\{c(\theta), y(\theta, c), \alpha(\theta)\}$ is incentive compatible if and only if the following four conditions hold:*
(i) Monotonicity:

The cutoff $c(\theta)$ is increasing in θ;

(ii) The second-stage envelope condition:

$$\tilde{\pi}(\theta, c, c) = \tilde{\pi}(\theta, \overline{c}, \overline{c}) + \max\{0, c(\theta) - c\}, \forall c, \forall\theta;$$

(iii) The first-stage envelope condition:

$$\pi(\theta, \theta) = \pi(\underline{\theta}, \underline{\theta}) + \int_{\underline{\theta}}^{\theta} \int_{\underline{c}}^{c(s)} H_2(c, s + \alpha(s)) dc ds, \ \forall\theta;$$

(iv) The moral hazard constraint: $\alpha(\theta)$ satisfies the moral hazard constraint MHC (10):

$$-C'(\alpha(\theta)) + \int_{\underline{c}}^{c(\theta)} H_2(c, \theta + \alpha(\theta)) dc \le 0, \textit{ with equality if } \alpha(\theta) > 0, \forall\theta.$$

The necessity of conditions (ii)-(iv) follows immediately by applying equation (7) and Lemma 1; the monotonicity of cutoffs in first-stage types is also intuitive. The significance of Lemma 2 is that these four conditions are not only necessary for IC, but also *sufficient.* This result will greatly facilitate our analysis, because it tells us whether a particular mechanism is indeed a valid candidate for the solution of the principal's problem.

4.4.2 Mitigation and Discussion

For the cutoff mechanism $\{c(\theta), y(\theta, c), \alpha(\theta)\}$, the total cost in (11) can be expressed as

$$\widetilde{TC} = \int_{\underline{\theta}}^{\overline{\theta}} \left\{ \begin{array}{c} C(\alpha(\theta)) + c_0 \\ + \int_{\underline{c}}^{c(\theta)} \left[c + \frac{1-G(\theta)}{g(\theta)} \frac{H_2(c,\theta+\alpha(\theta))}{h(c,\theta+\alpha(\theta))} - c_0 \right] h(c, \theta + \alpha(\theta)) dc \end{array} \right\} g(\theta) d\theta + \pi(\underline{\theta}, \underline{\theta}). \tag{16}$$

The principal's problem is

$$\min_{\{\alpha(\theta)\geq 0,\ c(\theta)\in[\underline{c},\overline{c}],\ y(\theta,c)\}} \widetilde{TC}, \text{ subject to conditions (i)-(iv) in Lemma 2 and IR constraint (13).}$$

Call this Problem (O-C), and denote the optimal solution to Problem (O-C) as $\{c^*(\theta), y^*(\theta, c), \alpha^*(\theta)\}$. In addition to Assumption 1, we make the following assumption for our analysis.

Assumption 2. $\frac{H_2(c,z)}{h(c,z)}$ *is decreasing in* $z \in [\underline{\theta}, +\infty)$.

Assumption 2 parallels Assumption 2 in Eső and Szentes [4]; moreover, together with increasing hazard rate $\frac{g(\theta)}{1-G(\theta)}$ and increasing $H_2(c, \theta)/h(c, \theta)$ in c (paralleling Assumption 1 in Eső and Szentes [4]), Assumption 2 implies the regularity assumption (Assumption 1). Note that Assumption 2 is automatically satisfied for the CDF $H(c, z) = 1 - (1 - F(c))^{z+\beta_0}$, as defined in (1).[8]

Finally, the following assumption guarantees strict mitigation, as will be reveled in Proposition 2.

Assumption 3 (Effective Investment Technology)**.**

$$C'(0) < \min_{\theta\in[\underline{\theta},\overline{\theta}]} \int_{\underline{c}}^{c_B(\theta)} H_2(c, \theta) dc.$$

Note that $\min_{\theta\in[\underline{\theta},\overline{\theta}]} \int_{\underline{c}}^{c_B(\theta)} H_2(c, \theta)dc$ is some fixed, strictly positive number, so Assumption 3 is satisfied when $C'(0)$ is small enough. Assumption 3 means that the marginal cost of investment is relatively small, which guarantees that all θ types have strong incentive to exert effort.

Equipped with Lemma 2, we have the following result, which formally states the phenomenon of mitigation of allocative distortion.

[8] Notice that $\frac{H_2(c,z)}{h(c,z)} = -\frac{(1-F(c))\ln(1-F(c))}{(z+\beta_0)f(c)}$, which is obvioulsy decreasing in $z \in [0, +\infty)$.

Proposition 2. *Under Assumptions 1 and 2, the type-enhancing investment mitigates allocative distortion; that is,* $c^*(\theta) \in [c_B(\theta), c_0]$*,* $\forall \theta \in [\underline{\theta}, \overline{\theta}]$*.*[9] *Moreover, if Assumption 3 is further imposed, then* $c^*(\theta) \in (c_B(\theta), c_0]$*,* $\forall \theta \in [\underline{\theta}, \overline{\theta})$*;*[10] *that is, the mitigation is strict.*

This mitigation of distortion can be understood as follows. When the second-stage allocative cutoff for type θ is $c^*(\theta)$, the principal excludes any second-stage type $c \in (c^*(\theta), \overline{c}]$ from trade. From (iv) of Lemma 2, a lower allocative cutoff $c^*(\theta)$—i.e., excluding more second-stage types from trade—leads to a lower investment. Therefore, doing so discourages the agent's incentive to invest. In particular, if $c^*(\theta) < c_B(\theta)$, all relatively efficient types $c \in (c^*(\theta), c_B(\theta)]$ are excluded. Excluding such sufficiently efficient types from trade not only dampens the agent's investment incentive, but also harms the principal. Therefore, the principal should mitigate the allocative distortion. On the other hand, it also hurts the principal if she overshoots in mitigating the allocative distortion by setting the cutoff $c^*(\theta)$ higher than c_0. This is because such an overshoot induces an inefficiently high investment, by trading with inefficient second-second types (i.e., types higher than c_0).

The investment-enhancing effect of not excluding sufficiently efficient types from trade can become dominant: With linear investment cost, the principal should not exclude any efficient second-stage type (i.e., types lower than c_0) from trade, so that ex post inefficiency disappears—i.e., the optimum is ex post efficient. The next section will further explore this.

4.5 The Linear-cost Model: Optimal Design Requires Social Efficiency

A special case of our general model is the linear-cost model with investment cost function $C(\alpha) = \gamma_0 \alpha$, where $\gamma_0 > 0$ is a constant. From (5), under Assumption 3, $\alpha^{FB}(\theta) > 0$ and satisfies $\int_{\underline{c}}^{c_0} H_2(c, \theta + \alpha^{FB}(\theta))dc = \gamma_0$. Let θ^* be the unique solution to $\int_{\underline{c}}^{c_0} H_2(c, \theta^*)dc = \gamma_0$; note that $\theta^* > \overline{\theta}$, implied by Assumption 3. Thus, $\alpha^{FB}(\theta) = \theta^* - \theta$.

As a direct corollary of Proposition 2, we obtain the following result for the linear-cost model.

Proposition 3. *For the linear-cost model, under Assumptions 1, 2, and 3, the optimal solution to Problem (O-C) satisfies*

$$c^*(\theta) = c_0, \ a^*(\theta) = a^{FB}(\theta) = \theta^* - \theta, \ \forall \theta.$$

[9] One caveat is that the two endpoints $\underline{\theta}$ and $\overline{\theta}$ are special, because a decrease (increase) in $c^*(\underline{\theta})$ ($c^*(\overline{\theta})$) can still lead to a feasible contract (since the new contract still satisfies all conditions in Lemma 2). However, changes made on just these two points will not change the function value $\widetilde{TC}$. Nevertheless, if we require that the solution $c^*(\theta)$ be, for example, continuous in θ, then this will rule out such a possibility.

[10] Similar to footnote 9, here $\underline{\theta}$ is special.

As a result, the cost-minimizing mechanism of the linear-cost model is ex post efficient and therefore coincides with the efficient mechanism. Moreover, for any first-stage type, there is no distortion of the induced investment, compared with the first-best level—i.e., the investment level is the same as the first-best one.[11]

A typical result from the sequential screening literature is that the second-stage allocation is discriminatory over first-stage types. However, here, by introducing first-stage hidden actions to sequential screening settings, the principal completely loses the power of dynamic screening, as the optimal contract responds only to the agent's second-stage type. In classical sequential screening settings, but with ex post participation constraints, Krähmer and Strausz [7] and Bergemann, Castro, and Weintraub [2] demonstrate that under some conditions, the optimal contract is a static one, which only screens the second-stage type. Our result suggests another possibility: With a first-stage type-enhancing hidden action, the optimal two-stage contract can also only screen the second-stage type, though the contract still needs to be implemented in two stages. Nevertheless, since equilibrium investment differs across first-stage types, their interim payoffs also vary.[12]

More importantly, the optimal contract is both ex ante and ex post efficient, so there is no efficiency loss. This is striking, since classical insights tell us that there should be efficiency loss (distortion of allocative efficiency or effort) to overcome information asymmetry.

As mentioned before, this result arises from the fact that the investment-enhancing effect of not excluding sufficiently efficient types from trade becomes dominant for the linear-cost case. At the optimum, the principal should not exclude any ex post efficient type from trade. It can be better understood when we focus on mechanisms that induce positive effort for all first-stage types. The principal's total cost is the sum of the social cost and the agent's information rent. Obviously, among all the feasible mechanisms (that induce positive effort for all first-stage types), the efficient mechanism minimizes the social cost—i.e., the principal should not exclude any ex post efficient type from trade. Thus, if the efficient mechanism also minimizes the agent's information rent, it must be optimal. However, for any incentive compatible mechanism that induces positive effort for all first-stage types, the moral hazard constraint MHC (10) is binding: $-\gamma_0+\int_{\underline{c}}^{\overline{c}} p(\theta,c)H_2(c,\theta+\alpha(\theta))dc = 0$.

[11] Proposition 3 can be strengthened. The optimal solution to Problem (O-C)—i.e., the efficient mechanism—is also the optimal solution to Problem (O). That is, the efficient mechanism is optimal in the class of stochastic mechanisms. Moreover, this result holds when only Assumptions 1 and 3 are imposed. The proof is available from the authors upon request.

[12] The first-stage expected payoff of type θ is $\gamma_0(\theta-\underline{\theta})$, as will be established later.

Lemma 1 further shows that the information rent $\pi(\theta, \theta)$ for type θ must satisfy

$$\pi(\theta, \theta) = \pi(\underline{\theta}, \underline{\theta}) + \int_{\underline{\theta}}^{\theta} \int_{\underline{c}}^{\overline{c}} p(s, c) H_2(c, s + \alpha(s)) dc ds = \pi(\underline{\theta}, \underline{\theta}) + \gamma_0 (\theta - \underline{\theta}) .$$

But this is exactly the information rent the first-best solution would give to type θ. Therefore, the efficient mechanism must be optimal.

5 Concluding Remarks

This paper introduces first-stage type-enhancing hidden investment to the standard sequential screening model. In the standard sequential screening model, the second-stage mechanism often involves discrimination against less efficient first-stage types. We find that with investment, the mitigation of allocative distortion must arise. Furthermore, such mitigation can even be extreme with linear investment cost: When the marginal cost of investment is small, the optimal mechanism maximizes social welfare, and the optimal mechanism only screens based on second-stage types. This result complements the conventional insight in the screening literature with pure adverse selection that efficiency must be sacrificed to reduce information rent at the optimum.

6 Appendix

Appendix A proves the results in the main text, and Appendix B provides proofs for the lemmas used in Appendix A.

6.1 Appendix A

Derivation of the moral hazard constraint (10): Since (7) holds for all reported type $\hat{\theta}$, we have

$$\begin{aligned}\hat{\pi}(\alpha, \hat{\theta}, \theta) &= -C(\alpha) + \int_{\underline{c}}^{\overline{c}} \left(\tilde{\pi}(\hat{\theta}, \overline{c}, \overline{c}) + \int_{c}^{\overline{c}} p(\hat{\theta}, s) ds \right) h(c, \theta + \alpha) dc \\ &= -C(\alpha) + \tilde{\pi}(\hat{\theta}, \overline{c}, \overline{c}) + \int_{\underline{c}}^{\overline{c}} p(\hat{\theta}, c) H(c, \theta + \alpha) dc. \end{aligned} \tag{17}$$

Taking derivative with respect to α yields[13]

$$\frac{\partial \hat{\pi}(\alpha, \hat{\theta}, \theta)}{\partial \alpha} = -C'(\alpha) + \int_{\underline{c}}^{\overline{c}} p(\hat{\theta}, c) H_2(c, \theta + \alpha) dc. \tag{18}$$

Second-order derivative

$$\frac{\partial^2 \hat{\pi}(\alpha, \hat{\theta}, \theta)}{\partial \alpha^2} = -C''(\alpha) + \int_{\underline{c}}^{\overline{c}} p(\hat{\theta}, c) H_{22}(c, \theta + \alpha) dc < 0,$$

when $p(\hat{\theta}, c) > 0$ on a positive measure subset of $[\underline{c}, \overline{c}]$, as $C'' \geq 0$. Since the agent's expected payoff $\hat{\pi}(\alpha, \hat{\theta}, \theta)$ is (strictly) concave in α, the optimal α—denoted as $\alpha(\hat{\theta}, \theta)$—is unique.[14] Thus, the agent with type θ who reports $\hat{\theta}$ will choose optimal effort level $\alpha(\hat{\theta}, \theta)$, characterized by

$$-C'(\alpha(\hat{\theta}, \theta)) + \int_{\underline{c}}^{\overline{c}} p(\hat{\theta}, c) H_2(c, \theta + \alpha(\hat{\theta}, \theta)) dc \leq 0, \text{ with equality if } \alpha(\hat{\theta}, \theta) > 0. \tag{19}$$

Now $\pi(\hat{\theta}, \theta)$—the type θ agent's highest expected payoff when he reports $\hat{\theta}$—can be expressed as

$$\pi(\hat{\theta}, \theta) = \hat{\pi}(\alpha(\hat{\theta}, \theta), \hat{\theta}, \theta) = -C(\alpha(\hat{\theta}, \theta)) + \tilde{\pi}(\hat{\theta}, \overline{c}, \overline{c}) + \int_{\underline{c}}^{\overline{c}} p(\hat{\theta}, c) H(c, \theta + \alpha(\hat{\theta}, \theta)) dc. \tag{20}$$

When the type θ agent reports truthfully, the principal's recommendation $\alpha(\theta)$ must coincide with the agent's optimal effort choice (obedience), so that $\alpha(\theta) = \alpha(\theta, \theta)$, leading to the *MHC* constraint (10). Since the agent's utility function $\hat{\pi}(\alpha, \hat{\theta}, \theta)$ is strictly concave in α (> 0), the "first-order approach" is valid; we can replace the original incentive compatibility constraint for moral hazard with the above first-order condition. □

Proof of Lemma 1: The moral hazard constraint is obvious. Note that $\alpha(\hat{\theta}, \theta)$ must be bounded, as $\lim_{\alpha \to +\infty} \hat{\pi}(\alpha, \hat{\theta}, \theta) = -\infty$ from (17). The following lemma establishes useful properties of $\alpha(\hat{\theta}, \theta)$ and $\pi(\hat{\theta}, \theta)$.

Lemma A1. *For any $\hat{\theta}$, $\alpha(\hat{\theta}, \cdot)$ is continuous. Moreover, it is differentiable everywhere except possibly at one point.*

Lemma A2. *For any $\hat{\theta}$, $\pi(\hat{\theta}, \cdot)$ is continuously differentiable; moreover, $\frac{\partial \pi(\hat{\theta}, \theta)}{\partial \theta} = \int_{\underline{c}}^{\overline{c}} p(\hat{\theta}, c) H_2(c, \theta + \alpha(\hat{\theta}, \theta)) dc$.*

[13] The differentiablity is implied by the Lebesgue's dominated convergence theorem.

[14] $\frac{\partial^2 \hat{\pi}(\alpha, \hat{\theta}, \theta)}{\partial \alpha^2} = 0$ only when $C'' = 0$ and $p(\hat{\theta}, c) = 0$ almost everywhere on $[\underline{c}, \overline{c}]$. However, in this case, from the first-order condition, the agent will optimally choose $\alpha = 0$. Therefore, in all cases the optimal α is unique. Note that $\alpha(\hat{\theta}, \theta)$ exists, as $\lim_{\alpha \to +\infty} \hat{\pi}(\alpha, \hat{\theta}, \theta) = -\infty$ from (17).

The proofs of Lemmas A1 and A2 are relegated to Appendix B. Since $\pi(\hat{\theta}, \theta)$ is continuously differentiable in θ over $[\underline{\theta}, \overline{\theta}]$ by Lemma A2, it is Lipschitz continuous and thus absolutely continuous. Also note that the derivative $\frac{\partial \pi(\hat{\theta},\theta)}{\partial \theta}$ is bounded by Lemma A2. By the envelope theorem (cf. Milgrom and Segal [10]), we have

$$\frac{d\pi(\theta,\theta)}{d\theta} = \frac{\partial \pi(\hat{\theta},\theta)}{\partial \theta}|_{\hat{\theta}=\theta} = \int_{\underline{c}}^{\overline{c}} p(\theta,c)H_2(c,\theta+\alpha(\theta))dc,$$

so that

$$\pi(\theta,\theta) = \pi(\underline{\theta},\underline{\theta}) + \int_{\underline{\theta}}^{\theta} \int_{\underline{c}}^{\overline{c}} p(s,c)H_2(c,s+\alpha(s))dcds. \tag{21}$$

□

Proof of Lemma 2: Necessity: As discussed after Lemma 2 in the main text, we only need to show the monotonicity condition (i) for this part. To this end, the following lemma is helpful.

Lemma A3. *Suppose $\theta_1 \le \theta_2$; then the function*

$$g(t) = C(\beta_1(t)) - C(\beta_2(t)) + \int_{\underline{c}}^{t} (H(c,\theta_2+\beta_2(t)) - H(c,\theta_1+\beta_1(t)))dc, \ t \in [\underline{c}, \overline{c}]$$

is increasing, where $\beta_i(t)$ is the unique maximizer to the function with respect to α defined on $[0,+\infty)$:

$$-C(\alpha) + \int_{\underline{c}}^{t} H(c,\theta_i+\alpha)dc, \ i=1,2.$$

That is,

$$-C'(\beta_i(t)) + \int_{\underline{c}}^{t} H_2(c,\theta_i+\beta_i(t))dc \le 0, \ \textit{with equality if } \beta_i(t)>0, \ i=1,2. \tag{22}$$

Now pick any θ and $\hat{\theta}$ with $\theta \leq \hat{\theta}$. From (20) and the first-stage IC, we have

$$\begin{aligned}
\pi(\hat{\theta},\theta) &= -C(\alpha(\hat{\theta},\theta)) + \tilde{\pi}(\hat{\theta},\bar{c},\bar{c}) + \int_{\underline{c}}^{c(\hat{\theta})} H(c,\theta+\alpha(\hat{\theta},\theta))dc \\
&= -C(\alpha(\hat{\theta},\hat{\theta})) + \tilde{\pi}(\hat{\theta},\bar{c},\bar{c}) + \int_{\underline{c}}^{c(\hat{\theta})} H(c,\hat{\theta}+\alpha(\hat{\theta},\hat{\theta}))dc + C(\alpha(\hat{\theta},\hat{\theta})) - \int_{\underline{c}}^{c(\hat{\theta})} H(c,\hat{\theta}+\alpha(\hat{\theta},\hat{\theta}))dc \\
&\quad -C(\alpha(\hat{\theta},\theta)) + \int_{\underline{c}}^{c(\hat{\theta})} H(c,\theta+\alpha(\hat{\theta},\theta))dc \\
&= \pi(\hat{\theta},\hat{\theta}) + C(\alpha(\hat{\theta},\hat{\theta})) - C(\alpha(\hat{\theta},\theta)) + \int_{\underline{c}}^{c(\hat{\theta})} (H(c,\theta+\alpha(\hat{\theta},\theta)) - H(c,\hat{\theta}+\alpha(\hat{\theta},\hat{\theta})))dc \\
&\leq \pi(\theta,\theta). \qquad (23)
\end{aligned}$$

Similarly,

$$\begin{aligned}
\pi(\theta,\hat{\theta}) &= \tilde{\pi}(\theta,\bar{c},\bar{c}) - C(\alpha(\theta,\hat{\theta})) + \int_{\underline{c}}^{c(\theta)} H(c,\hat{\theta}+\alpha(\theta,\hat{\theta}))dc \\
&= \tilde{\pi}(\theta,\bar{c},\bar{c}) - C(\alpha(\theta,\theta)) + \int_{\underline{c}}^{c(\theta)} H(c,\theta+\alpha(\theta,\theta))dc + C(\alpha(\theta,\theta)) - \int_{\underline{c}}^{c(\theta)} H(c,\theta+\alpha(\theta,\theta))dc \\
&\quad -C(\alpha(\theta,\hat{\theta})) + \int_{\underline{c}}^{c(\theta)} H(c,\hat{\theta}+\alpha(\theta,\hat{\theta}))dc \\
&= \pi(\theta,\theta) + C(\alpha(\theta,\theta)) - C(\alpha(\theta,\hat{\theta})) + \int_{\underline{c}}^{c(\theta)} (H(c,\hat{\theta}+\alpha(\theta,\hat{\theta})) - H(c,\theta+\alpha(\theta,\theta)))dc \\
&\leq \pi(\hat{\theta},\hat{\theta}). \qquad (24)
\end{aligned}$$

Combining (23) and (24), we have

$$\begin{aligned}
& C(\alpha(\theta,\theta)) - C(\alpha(\theta,\hat{\theta})) + \int_{\underline{c}}^{c(\theta)} (H(c,\hat{\theta}+\alpha(\theta,\hat{\theta})) - H(c,\theta+\alpha(\theta,\theta)))dc \\
\leq\ & C(\alpha(\hat{\theta},\theta)) - C(\alpha(\hat{\theta},\hat{\theta})) + \int_{\underline{c}}^{c(\hat{\theta})} (H(c,\hat{\theta}+\alpha(\hat{\theta},\hat{\theta})) - H(c,\theta+\alpha(\hat{\theta},\theta)))dc.
\end{aligned}$$

According to Lemma A3, the above inequality is simply $g(c(\theta)) \leq g(c(\hat{\theta}))$, which holds if and only if $c(\theta) \leq c(\hat{\theta})$. This completes the proof of necessity.

Sufficiency: First, condition (ii) is simply the envelope condition (7) for cutoff mechanisms.

Moreover, cutoff mechanisms, by definition, automatically satisfy the second-stage monotonicity condition—i.e., $p(\theta, c)$ is decreasing in c. Thus, for cutoff mechanisms, condition (ii) implies the second-stage IC.

Now it remains to show that conditions (i)-(iv) imply the first-stage IC. Notice that $\alpha(\hat{\theta}, \theta)$ defined in (9) still applies, as it only uses the second-stage envelope condition (condition (ii) here) and talks about the type θ agent's optimal investment level when he reports $\hat{\theta}$; it has nothing to do with the first-stage IC. Therefore, condition (iv) is then simply the obedience constraint, as required by the first-stage IC: $\alpha(\theta) = \alpha(\theta, \theta)$ for all θ—that is, the principal's effort recommendation $\alpha(\theta)$ must be the same as the type θ agent's optimal investment level $\alpha(\theta, \theta)$ when he reports his type truthfully.

For any type θ and any report $\hat{\theta}$, define $\pi(\hat{\theta}, \theta)$ as it is in (20). Now pick any type θ. We only need to show that $\pi(\hat{\theta}, \theta) \leq \pi(\theta, \theta)$ holds for any $\hat{\theta}$ under conditions (i)-(iv). Similar to (23), $\pi(\hat{\theta}, \theta) \leq \pi(\theta, \theta)$ is equivalent to showing that

$$C(\alpha(\hat{\theta}, \hat{\theta})) - C(\alpha(\hat{\theta}, \theta)) + \int_{\underline{c}}^{c(\hat{\theta})} (H(c, \theta + \alpha(\hat{\theta}, \theta)) - H(c, \hat{\theta} + \alpha(\hat{\theta}, \hat{\theta})))dc \leq \pi(\theta, \theta) - \pi(\hat{\theta}, \hat{\theta}). \quad (25)$$

Our goal is to show that the above inequality (25) holds under conditions (i)-(iv).

We show this for the case that $\hat{\theta} \geq \theta$; the case that $\hat{\theta} < \theta$ is similar. Notice that Lemmas A1 and A2 and arguments in their proofs still apply here, as they only use the second-stage envelope condition (condition (ii) here) and do not rely on the first-stage IC. Then,

$$C(\alpha(\hat{\theta}, \hat{\theta})) - C(\alpha(\hat{\theta}, \theta)) = \int_{\theta}^{\hat{\theta}} C'(\alpha(\hat{\theta}, s)) \frac{\partial \alpha(\hat{\theta}, s)}{\partial s} ds.$$

Similarly,

$$\int_{\underline{c}}^{c(\hat{\theta})} (H(c, \theta + \alpha(\hat{\theta}, \theta)) - H(c, \hat{\theta} + \alpha(\hat{\theta}, \hat{\theta})))dc = -\int_{\theta}^{\hat{\theta}} \int_{\underline{c}}^{c(\hat{\theta})} H_2(c, s + \alpha(\hat{\theta}, s)) \left(1 + \frac{\partial \alpha(\hat{\theta}, s)}{\partial s}\right) dcds.$$

Therefore, the LHS of (25) can be written as

$$\int_{\theta}^{\hat{\theta}} \left[\frac{\partial \alpha(\hat{\theta}, s)}{\partial s} \left(C'(\alpha(\hat{\theta}, s)) - \int_{\underline{c}}^{c(\hat{\theta})} H_2(c, s + \alpha(\hat{\theta}, s))dc \right) - \int_{\underline{c}}^{c(\hat{\theta})} H_2(c, s + \alpha(\hat{\theta}, s)) \right] ds.$$

From (34) in the proof of Lemma A2, the above expression can be simplified as

$$-\int_{\theta}^{\hat{\theta}} \int_{\underline{c}}^{c(\hat{\theta})} H_2(c, s + \alpha(\hat{\theta}, s))ds.$$

By condition (iii) and $\alpha(s) = \alpha(s, s)$ (by condition (iv)), the RHS of (25) is

$$-\int_{\theta}^{\hat{\theta}} \int_{\underline{c}}^{c(s)} H_2(c, s + \alpha(s))dcds = -\int_{\theta}^{\hat{\theta}} \int_{\underline{c}}^{c(s)} H_2(c, s + \alpha(s, s))dcds.$$

Therefore, to show IC, it suffices to show that

$$\int_{\underline{c}}^{c(\hat{\theta})} H_2(c, s + \alpha(\hat{\theta}, s))dc \geq \int_{\underline{c}}^{c(s)} H_2(c, s + \alpha(s, s))dc, \text{ for any } s \in [\theta, \hat{\theta}]. \tag{26}$$

Notice that $c(s) \leq c(\hat{\theta})$ for all $s \in [\theta, \hat{\theta}]$ by condition (i). Recall from (9) the definition of $\alpha(\hat{\theta}, s)$:

$$-C'(\alpha(\hat{\theta}, s)) + \int_{\underline{c}}^{c(\hat{\theta})} H_2(c, s + \alpha(\hat{\theta}, s))dc \leq 0, \text{ with equality if } \alpha(\hat{\theta}, s) > 0,$$

$$-C'(\alpha(s, s)) + \int_{\underline{c}}^{c(s)} H_2(c, s + \alpha(s, s))dc \leq 0, \text{ with equality if } \alpha(s, s) > 0.$$

Note that $\alpha(s, s) \leq \alpha(\hat{\theta}, s)$ for any $s \in [\theta, \hat{\theta}]$. This is because if $\alpha(\hat{s}, \hat{s}) > \alpha(\hat{\theta}, \hat{s}) (\geq 0)$ for some $\hat{s} \in [\theta, \hat{\theta}]$, then

$$\int_{\underline{c}}^{c(\hat{s})} H_2(c, \hat{s} + \alpha(\hat{s}, \hat{s}))dc = C'(\alpha(\hat{s}, \hat{s})) \geq C'(\alpha(\hat{\theta}, \hat{s})) \geq \int_{\underline{c}}^{c(\hat{\theta})} H_2(c, \hat{s} + \alpha(\hat{\theta}, \hat{s}))dc.$$

However, if $\alpha(\hat{s}, \hat{s}) > \alpha(\hat{\theta}, \hat{s})$, then

$$\int_{\underline{c}}^{c(\hat{s})} H_2(c, \hat{s} + \alpha(\hat{s}, \hat{s}))dc < \int_{\underline{c}}^{c(\hat{\theta})} H_2(c, \hat{s} + \alpha(\hat{\theta}, \hat{s}))dc,$$

which is a contradiction.

Therefore, $\alpha(s, s) \leq \alpha(\hat{\theta}, s)$ for any $s \in [\theta, \hat{\theta}]$. Thus, when $\alpha(\hat{\theta}, s) = 0$, $\alpha(s, s) = 0$ as well, so (26) holds as $c(s) \leq c(\hat{\theta})$. When $\alpha(\hat{\theta}, s) > 0$,

$$\int_{\underline{c}}^{c(\hat{\theta})} H_2(c, s + \alpha(\hat{\theta}, s))dc = C'(\alpha(\hat{\theta}, s)) \geq C'(\alpha(s, s)) \geq \int_{\underline{c}}^{c(s)} H_2(c, s + \alpha(s, s))dc,$$

which implies that (26) holds. This completes the proof of sufficiency. $\square$

Proof of Proposition 2: From condition (iii) of Lemma 2, the IR constraint (13) is equivalent to $\pi(\underline{\theta}, \underline{\theta}) \geq 0$, which must be binding at the optimum. Dropping further conditions (ii) and (iii) of Lemma 2, Problem (O-C) can be relaxed to the following Problem (O-C-R).

$$\min_{\{\alpha(\theta)\geq 0,\ c(\theta)\in[\underline{c},\overline{c}]\}} \int_{\underline{\theta}}^{\overline{\theta}} \left\{ \begin{array}{c} C(\alpha(\theta)) + c_0 \\ + \int_{\underline{c}}^{c(\theta)} \left[c + \frac{1-G(\theta)}{g(\theta)} \frac{H_2(c,\theta+\alpha(\theta))}{h(c,\theta+\alpha(\theta))} - c_0\right] h(c, \theta + \alpha(\theta))dc \end{array} \right\} g(\theta)d\theta$$

subject to

$$c(\theta) \text{ is increasing in } \theta; \tag{27}$$

$$-C'(\alpha(\theta)) + \int_{\underline{c}}^{c(\theta)} H_2(c, \theta + \alpha(\theta))dc \leq 0, \text{ with equality if } \alpha(\theta) > 0,\ \forall \theta. \tag{28}$$

The solution of Problem (O-C-R) cannot be worse than Problem (O-C).

Dropping (27), Problem (O-C-R) is further relaxed to Problem (O-C-R-R), which is a pointwise problem, as Problem (O-C-R-R) does not involve any restriction between two different θ's. For type θ, when the cutoff is $\hat{c}$, the investment α is uniquely pinned down by (28):

$$-C'(\alpha) + \int_{\underline{c}}^{\hat{c}} H_2(c, \theta + \alpha)dc \leq 0, \text{ with equality if } \alpha > 0.$$

It is easy to see that α is increasing in $\hat{c}$ and is strictly increasing when $\alpha > 0$. Therefore, we denote α as $\alpha(\hat{c})$, as a function of the cutoff $\hat{c}$. Define the single-variable function $\varphi(\cdot; \theta)$ as

$$\varphi(\hat{c}; \theta) = C(\alpha(\hat{c})) + \int_{\underline{c}}^{\hat{c}} \left[c + \frac{1 - G(\theta)}{g(\theta)} \frac{H_2(c, \theta + \alpha(\hat{c}))}{h(c, \theta + \alpha(\hat{c}))} - c_0\right] h(c, \theta + \alpha(\hat{c}))dc + c_0,\ \hat{c} \in [\underline{c}, \overline{c}],$$

where

$$-C'(\alpha(\hat{c})) + \int_{\underline{c}}^{\hat{c}} H_2(c, \theta + \alpha(\hat{c}))dc \leq 0, \text{ with equality if } \alpha(\hat{c}) > 0. \tag{29}$$

For type θ, Problem (O-C-R-R) can be expressed as

$$\min_{\hat{c}\in[\underline{c},\overline{c}]} \varphi(\hat{c}; \theta).$$

The following result characterizes properties of the optimal solution to Problem (O-C-R-R).

Lemma A4. *Under Assumptions 1 and 2, for Problem (O-C-R-R), $\varphi(\hat{c}; \theta) > \varphi(c_B(\theta); \theta)$ when*

$\hat{c} < c_B(\theta)$, and $\varphi(\hat{c};\theta) > \varphi(c_0;\theta)$ when $\hat{c} > c_0$. If Assumption 3 is further imposed, then for any $\theta < \overline{\theta}$, there exists some $\hat{c}_1 \in (c_B(\theta), c_0]$ such that $\varphi(c_B(\theta);\theta) > \varphi(\hat{c}_1;\theta)$.

A direct implication of Lemma A4 is that $\varphi(\hat{c};\theta)$ attains its minimum when $\hat{c} \in [c_B(\theta), c_0]$, under Assumptions 1 and 2. Now we are ready to show the first statement of Proposition 2. Suppose that $\{c^*(\theta), \alpha^*(\theta), y^*(\theta, c)\}$ is the optimal solution to Problem (O-C). Note that since $c^*(\theta)$ is the solution to Problem (O-C), it must satisfy the monotonicity constraint (condition (i)) in Lemma 2.

Define the following three sets: $L_1 = \{\theta \in [\underline{\theta}, \overline{\theta}] : c^*(\theta) < c_B(\theta)\}$; $L_2 = \{\theta \in [\underline{\theta}, \overline{\theta}] : c^*(\theta) \in [c_B(\theta), c_0]\}$; and $L_3 = \{\theta \in [\underline{\theta}, \overline{\theta}] : c^*(\theta) > c_0\}$. Obviously, $L_1 \cup L_2 \cup L_3 = [\underline{\theta}, \overline{\theta}]$. Since $c^*(\theta)$ is increasing, L_3 is connected and every element in $L_1 \cup L_2$ is lower than an element in L_3.

It suffices to show that $L_1 = L_3 = \emptyset$ for Proposition 2. As mentioned in footnote 9, what we need to show is $L_1 \cap [\underline{\theta}, \overline{\theta}] \subseteq \{\underline{\theta}\}$ and $L_3 \cap [\underline{\theta}, \overline{\theta}] \subseteq \{\overline{\theta}\}$. The proof consists of two steps.

Step 1: Consider the following cutoff $c^{**}(\theta)$ for $\theta \in [\underline{\theta}, \overline{\theta}]$:

$$c^{**}(\theta) = \begin{cases} c_B(\theta), \text{ if } \theta \in L_1; \\ c^*(\theta), \text{ if } \theta \in L_2; \\ c_0, \text{ if } \theta \in L_3. \end{cases}$$

We first show that $c^{**}(\theta)$ also satisfies the monotonicity constraint (condition (i)) in Lemma 2. For this purpose, we only need to show that $c^{**}(\theta)$ is increasing in $L_1 \cup L_2$. Notice that in $L_1 \cup L_2$, we have $c^{**}(\theta) = \max\{c_B(\theta), c^*(\theta)\}$. Since both $c_B(\theta)$ and $c^*(\theta)$ are increasing, it is clear that $\max\{c_B(\theta), c^*(\theta)\}$ must be increasing in $L_1 \cup L_2$.

Since $c^{**}(\theta)$ satisfies the monotonicity constraint (condition (i)) in Lemma 2, based on $c^{**}(\theta)$, one can construct investment $a^{**}(\theta)$ and payment rule $y^{**}(\theta, c)$ with $\pi(\underline{\theta}, \underline{\theta}) = 0$, following conditions (ii)–(iv) in Lemma 2. Then, by construction, the cutoff mechanism $\{c^{**}(\theta), a^{**}(\theta), y^{**}(\theta, c)\}$ satisfies all constraints of Problem (O-C), so that it is feasible in Problem (O-C).

Step 2: Now we are ready to show that $L_1 \cap [\underline{\theta}, \overline{\theta}] \subseteq \{\underline{\theta}\}$ and $L_3 \cap [\underline{\theta}, \overline{\theta}] \subseteq \{\overline{\theta}\}$. In fact, suppose, to the contrary, that either $L_1 \cap [\underline{\theta}, \overline{\theta}] \nsubseteq \{\underline{\theta}\}$ or $L_3 \cap [\underline{\theta}, \overline{\theta}] \nsubseteq \{\overline{\theta}\}$. We will argue that none of the following three cases is possible. We first argue that $L_1 \cap [\underline{\theta}, \overline{\theta}] \nsubseteq \{\underline{\theta}\}$ and $L_3 \cap [\underline{\theta}, \overline{\theta}] \subseteq \{\overline{\theta}\}$ is impossible; the arguments for $L_1 \cap [\underline{\theta}, \overline{\theta}] \subseteq \{\underline{\theta}\}$ and $L_3 \cap [\underline{\theta}, \overline{\theta}] \nsubseteq \{\overline{\theta}\}$, or $L_1 \cap [\underline{\theta}, \overline{\theta}] \nsubseteq \{\underline{\theta}\}$ and $L_3 \cap [\underline{\theta}, \overline{\theta}] \nsubseteq \{\overline{\theta}\}$, are similar, as can be easily seen below.

If $L_1 \cap [\underline{\theta}, \overline{\theta}] \nsubseteq \{\underline{\theta}\}$, there exists some $\theta_1 \in L_1$, where $\theta_1 \in (\underline{\theta}, \overline{\theta}]$. We claim that there exists some $\delta \in (0,\, \theta_1 - \underline{\theta}]$ such that $[\theta_1 - \delta, \theta_1] \subseteq L_1$. In fact, since $\theta_1 \in L_1$, $c^*(\theta_1) < c_B(\theta_1)$. Denote $\varepsilon = c_B(\theta_1) - c^*(\theta_1) > 0$. By the definition of $c_B(\theta)$, $c_B(\theta)$ is increasing and continuous in θ. Therefore, there exists a $\delta \in (0,\, \theta_1 - \underline{\theta}]$ such that $c_B(\theta_1) - c_B(\theta) = |c_B(\theta_1) - c_B(\theta)| < \varepsilon$ for all $\theta \in [\theta_1 - \delta, \theta_1]$. Then, for any $\theta \in [\theta_1 - \delta, \theta_1]$,

$$c_B(\theta) - c^*(\theta) > (c_B(\theta_1) - \varepsilon) - c^*(\theta) \geq (c_B(\theta_1) - \varepsilon) - c^*(\theta_1) = 0,$$

where the second inequality uses the fact that $c^*(\theta) \leq c^*(\theta_1)$. Therefore, for any $\theta \in [\theta_1 - \delta, \theta_1]$, $c_B(\theta) - c^*(\theta) > 0$, which means $[\theta_1 - \delta, \theta_1] \subseteq L_1$.

For any $\hat{\theta} \in [\theta_1 - \delta, \theta_1]$, since $c^*(\hat{\theta}) < c_B(\hat{\theta})$, $\varphi(c^{**}(\hat{\theta}); \hat{\theta}) < \varphi(c^*(\hat{\theta}); \hat{\theta})$ by Lemma A4. Note that the objective function of Problem (O-C) can be expressed as $\int_{\underline{\theta}}^{\overline{\theta}} \varphi(c(\theta); \theta) g(\theta) d\theta$. Therefore, changing from $\{c^*(\theta), \alpha^*(\theta), y^*(\theta, c)\}$ to $\{c^{**}(\theta), a^{**}(\theta), y^{**}(\theta, c)\}$, the objective function's value strictly decreases on the interval $[\theta_1 - \delta, \theta_1]$, whose measure is strictly positive. However, since $\{c^{**}(\theta), a^{**}(\theta), y^{**}(\theta, c)\}$ is feasible in Problem (O-C), this further implies that $\{c^{**}(\theta), a^{**}(\theta), y^{**}(\theta, c)\}$ leads to a strictly lower total procurement cost than $\{c^*(\theta), \alpha^*(\theta), y^*(\theta, c)\}$ does in Problem (O-C), which contradicts the fact that $\{c^*(\theta), \alpha^*(\theta), y^*(\theta, c)\}$ is the solution to Problem (O-C).

Now we proceed to the second statement of Proposition 2. As mentioned in Remark 1 in the proof of Lemma A4, under Assumptions 1, 2, and 3, for any fixed $\theta < \overline{\theta}$, $\varphi(\hat{c}; \theta)$ is strictly decreasing in $\hat{c} \in [c_B(\theta), \hat{c}^*(\theta)]$, where $\hat{c}^*(\theta) \in (c_B(\theta), c_0]$ is the smallest solution to $\frac{d\varphi(\hat{c};\theta)}{d\hat{c}} = 0$ on $\hat{c} \in [c_B(\theta), c_0]$. Define $\eta(\theta) = \hat{c}^*(\theta) - c_B(\theta)$. Then $\eta(\theta) > 0$ for any $\theta \in [\underline{\theta}, \overline{\theta})$. In Remark 1 there, we also introduce a function $\rho(\hat{c}, \theta)$, which is defined as $\rho(\hat{c}, \theta) = \frac{d\varphi(\hat{c};\theta)}{d\hat{c}}$; as mentioned there, $\rho(\hat{c}, \theta)$ is continuous in $(\hat{c}, \theta) \in \{(\hat{c}, \theta) : c_B(\theta) \leq \hat{c} \leq c_0,\ \underline{\theta} \leq \theta < \overline{\theta}]$. Thus, by definition, $\rho(\hat{c}^*(\theta), \theta) = 0$ for any $\theta \in [\underline{\theta}, \overline{\theta})$, and

$$\rho(\hat{c}, \theta) < 0, \text{ for any } (\hat{c}, \theta) \in \{(\hat{c}, \theta) : c_B(\theta) \leq \hat{c} < \hat{c}^*(\theta),\ \underline{\theta} \leq \theta < \overline{\theta}]. \tag{30}$$

Suppose that $\{c^*(\theta), \alpha^*(\theta), y^*(\theta, c)\}$ is the optimal solution to Problem (O-C). It then follows that $c^*(\theta)$ is increasing in θ. Notice first that $c^*(\theta) \in [c_B(\theta), c_0]$ for all $\theta \in (\underline{\theta}, \overline{\theta})$, by the first statement of Proposition 2 we just proved. The proof of strict mitigation consists of the following four steps.

Step 1: Let $J = [a, b]$ be a closed interval with $a \geq \underline{\theta}$ and $b < \overline{\theta}$. Define

$$\tilde{\eta}(J) = \inf_{\theta \in J} \eta(\theta). \tag{31}$$

Obviously, $\tilde{\eta}(J) \geq 0$. We claim that $\tilde{\eta}(J) > 0$. Suppose, to the contrary, that $\tilde{\eta}(J) = 0$. Then for any integer $n \geq 1$, there exists some $\theta_n \in J$, such that $\eta(\theta_n) \in (0, 1/n)$. This then defines a convergent sequence $\{\eta(\theta_n)\}_{n=1}^{\infty}$ whose limit is 0. Since J is compact, there exists a convergent subsequence $\{\theta_{n_k}\}_{k=1}^{\infty}$ of the sequence $\{\theta_n\}_{n=1}^{\infty}$ whose limit belongs to J. Denote $\theta_0 = \lim_{k\to\infty} \theta_{n_k}$. Then $\theta_0 \in J$. Notice that

$$0 = \lim_{k\to\infty} \eta(\theta_{n_k}) = \lim_{k\to\infty} (\hat{c}^*(\theta_{n_k}) - c_B(\theta_{n_k})) = \lim_{k\to\infty} \hat{c}^*(\theta_{n_k}) - \lim_{k\to\infty} c_B(\theta_{n_k}) = \lim_{k\to\infty} \hat{c}^*(\theta_{n_k}) - c_B(\theta_0),$$

where the last equality follows from the continuity of the function $c_B(\theta)$ in θ. Thus, $\lim_{k\to\infty} \hat{c}^*(\theta_{n_k}) = c_B(\theta_0)$. Let $\hat{c}_{n_k} = \hat{c}^*(\theta_{n_k})$. Then $\{\hat{c}_{n_k}\}_{k=1}^{\infty}$ is a convergent sequence whose limit is $c_B(\theta_0)$. Furthermore, for any integer $k \geq 1$, $\rho(\hat{c}_{n_k}, \theta_{n_k}) = 0$, by definition. Notice that $\lim_{k\to\infty}(\hat{c}_{n_k}, \theta_{n_k}) = (c_B(\theta_0), \theta_0)$. Then, since $\rho(\hat{c}, \theta)$ is continuous in $(\hat{c}, \theta) \in \{(\hat{c}, \theta) : c_B(\theta) \leq \hat{c} \leq c_0,\ \underline{\theta} \leq \theta < \overline{\theta}]$, we have

$$\lim_{k\to\infty} \rho(\hat{c}_{n_k}, \theta_{n_k}) = \rho(c_B(\theta_0), \theta_0) < 0,$$

where the inequality follows from (30). However, this contradicts $\rho(\hat{c}_{n_k}, \theta_{n_k}) = 0$ for all $k \geq 1$.

Step 2: We claim that for any $\varepsilon > 0$, there must exist some $\theta \in [\overline{\theta} - \varepsilon, \overline{\theta}]$ such that $c^*(\theta) > c_B(\theta)$. To see why this is true, suppose, to the contrary, that the claim is false. Then there is some $\varepsilon_0 > 0$, such that $c^*(\theta) = c_B(\theta)$, for all $\theta \in [\overline{\theta} - \varepsilon_0, \overline{\theta}]$ (without loss of generality, let ε_0 be relatively small such that $\overline{\theta} - \varepsilon_0 \geq \underline{\theta}$). Consider the interval $J' = [\overline{\theta} - \varepsilon_0, \overline{\theta} - \frac{\varepsilon_0}{2}]$. Let $\tilde{\eta}(J') = \inf_{\theta \in J'} \eta(\theta)$. By the argument above, $\tilde{\eta}(J') > 0$.

Construct a new cutoff $c^{**}(\theta)$ (we abuse the notation of c^{**} a bit):

$$c^{**}(\theta) = \begin{cases} c^*(\theta), \text{ if } \theta \in [\underline{\theta}, \overline{\theta} - \varepsilon_0); \\ \min\{c_B(\overline{\theta} - \frac{\varepsilon_0}{2}),\ c^*(\theta) + \tilde{\eta}(J')\}, \text{ if } \theta \in [\overline{\theta} - \varepsilon_0, \overline{\theta} - \frac{\varepsilon_0}{2}]; \\ c^*(\theta), \text{ if } \theta \in (\overline{\theta} - \frac{\varepsilon_0}{2}, \overline{\theta}]. \end{cases}$$

Let us check that $c^{**}(\theta)$ is increasing in θ. For any $\theta \in [\overline{\theta} - \varepsilon_0, \overline{\theta} - \frac{\varepsilon_0}{2}]$ and any $\theta' \in [\underline{\theta}, \overline{\theta} - \varepsilon_0)$,

$$\begin{aligned} c^{**}(\theta) &= \min\{c_B(\overline{\theta} - \frac{\varepsilon_0}{2}),\ c^*(\theta) + \tilde{\eta}(J')\} = \min\{c_B(\overline{\theta} - \frac{\varepsilon_0}{2}),\ c_B(\theta) + \tilde{\eta}(J')\} \\ &\geq c_B(\theta) = c^*(\theta) \geq c^*(\theta') = c^{**}(\theta'). \end{aligned}$$

$c^{**}(\theta)$ is obviously increasing in $\theta \in [\overline{\theta} - \varepsilon_0, \overline{\theta} - \frac{\varepsilon_0}{2}]$. And when $\theta \in (\overline{\theta} - \frac{\varepsilon_0}{2}, \overline{\theta}]$,

$$c^{**}(\theta) = c^*(\theta) = c_B(\theta) \geq c_B(\overline{\theta} - \frac{\varepsilon_0}{2}) = c^{**}(\overline{\theta} - \frac{\varepsilon_0}{2}).$$

Thus, $c^{**}(\theta)$ is increasing in $\theta \in [\underline{\theta}, \overline{\theta}]$.

Similar to the argument for the first statement of Proposition 2, since $c^{**}(\theta)$ satisfies the monotonicity constraint (condition (i)) in Lemma 2, based on the cutoff $c^{**}(\theta)$, one can then construct the investment $a^{**}(\theta)$ and payment rule $y^{**}(\theta, c)$ with $\pi(\underline{\theta}, \underline{\theta}) = 0$, according to conditions (ii)–(iv) in Lemma 2. Then, by construction, the cutoff mechanism $\{c^{**}(\theta), a^{**}(\theta), y^{**}(\theta, c)\}$ satisfies all constraints of Problem (O-C), so that it is feasible in Problem (O-C).

Recall the definition of $\tilde{\eta}(J')$. It implies that $\varphi(\hat{c}; \theta)$ is strictly decreasing in $\hat{c} \in [c_B(\theta), c_B(\theta) + \tilde{\eta}(J')]$, for all $\theta \in J'$. By construction, $c^{**}(\theta) \in [c_B(\theta), c_B(\theta) + \eta(J')]$, for all $\theta \in J'$. Therefore, for any $\theta \in J'$, $\varphi(c^{**}(\theta); \theta) < \varphi(c_B(\theta); \theta) = \varphi(c^*(\theta); \theta)$.

Notice that the objective function in Problem (O-C) can be expressed as $\int_{\underline{\theta}}^{\overline{\theta}} \varphi(c(\theta); \theta) g(\theta) d\theta$. Then,

$$\int_{\underline{\theta}}^{\overline{\theta}} \varphi(c^{**}(\theta); \theta) g(\theta) d\theta - \int_{\underline{\theta}}^{\overline{\theta}} \varphi(c^{**}(\theta); \theta) g(\theta) d\theta = \int_{\overline{\theta} - \varepsilon_0}^{\overline{\theta} - \frac{\varepsilon_0}{2}} \left[\varphi(c^{**}(\theta); \theta) - \varphi(c^*(\theta); \theta)\right] g(\theta) d\theta < 0,$$

which contradicts the optimality of $\{c^*(\theta), \alpha^*(\theta), y^*(\theta, c)\}$.

<u>Step 3</u>: Given step 2, for any $\varepsilon > 0$, there exists some $\theta_\varepsilon \in [\overline{\theta} - \varepsilon, \overline{\theta}]$ such that $c^*(\theta_\varepsilon) > c_B(\theta_\varepsilon)$. Denote the interval $[\underline{\theta}, \theta_\varepsilon]$ as K_ε. Obviously, $\theta_\varepsilon \to \overline{\theta}$, as $\varepsilon \to 0$. Let $\eta_\varepsilon = \min\{\tilde{\eta}(K_\varepsilon),\ c^*(\theta_\varepsilon) - c_B(\theta_\varepsilon)\} > 0$, where $\tilde{\eta}(K_\varepsilon) = \inf_{\theta \in K_\varepsilon} \eta(\theta) > 0$ as shown in step 1. We will show that $c^*(\theta) \in (c_B(\theta), c_0]$ for any $\theta \in K_\varepsilon \backslash \{\underline{\theta}\}$.

Notice first that $c^*(\theta) \in [c_B(\theta), c_0]$ for all $\theta \in (\underline{\theta}, \overline{\theta})$, by the first statement of Proposition 2 we just proved. Now suppose, to the contrary, that $c^*(\theta_1) = c_B(\theta_1)$ for some $\theta_1 \in K_\varepsilon \backslash \{\underline{\theta}\}$. Then, since $c_B(\theta)$ is continuous in θ and since $c^*(\theta)$ is increasing in θ, there exists some $\delta_1 > 0$, such that for any $\theta \in [\theta_1 - \delta_1, \theta_1] \subseteq K_\varepsilon$, we have $c_B(\theta_1) - c_B(\theta) < \eta_\varepsilon$. This implies that for any $\theta \in [\theta_1 - \delta_1, \theta_1]$,

$$|c^*(\theta) - c_B(\theta)| = c^*(\theta) - c_B(\theta) \leq c^*(\theta_1) - c_B(\theta) = c_B(\theta_1) - c_B(\theta) < \eta_\varepsilon. \tag{32}$$

Consider the following cutoff $c^{**}(\theta)$ for $\theta \in [\underline{\theta}, \overline{\theta}]$:

$$c^{**}(\theta) = \begin{cases} \max\{c^*(\theta),\ c_B(\theta) + \eta_\varepsilon\}, \text{ if } \theta \in [\underline{\theta}, \theta_\varepsilon]; \\ c^*(\theta), \text{ if } \theta \in (\theta_\varepsilon, \overline{\theta}]. \end{cases}$$

The idea of the construction of $c^{**}(\theta)$ in $\theta \in [\underline{\theta}, \theta_\varepsilon]$ is that if $c^*(\theta) - c_B(\theta) < \eta_\varepsilon$, then move $c^*(\theta)$ further to the level of $c_B(\theta) + \eta_\varepsilon$; otherwise, stick to the original cutoff $c^*(\theta)$. Let us check that $c^{**}(\theta)$ is increasing in $\theta \in [\underline{\theta}, \overline{\theta}]$. It is easy to see that $c^{**}(\theta)$ is increasing in $\theta \in [\underline{\theta}, \theta_\varepsilon]$, because both $c^*(\theta)$ and $c_B(\theta) + \eta_\varepsilon$ are increasing in θ. Thus, it suffices to check that $c^{**}(\theta_\varepsilon) < c^*(\theta)$, for any $\theta \in (\theta_\varepsilon, \overline{\theta}]$. To see this, notice that since $\eta_\varepsilon = \min\{\tilde{\eta}(K_\varepsilon),\ c^*(\theta_\varepsilon) - c_B(\theta_\varepsilon)]$, $c^*(\theta_\varepsilon) - c_B(\theta_\varepsilon) \geq \eta_\varepsilon$. It then follows that

$$c^{**}(\theta_\varepsilon) = \max\{c^*(\theta_\varepsilon),\ c_B(\theta_\varepsilon) + \eta_\varepsilon\} = c^*(\theta_\varepsilon).$$

Therefore, for any $\theta \in (\theta_\varepsilon, \overline{\theta}]$, $c^{**}(\theta_\varepsilon) = c^*(\theta_\varepsilon) < c^*(\theta)$.

Thus, similar to the argument for the first statement of Proposition 2, since $c^{**}(\theta)$ satisfies the monotonicity constraint (condition (i)) in Lemma 2, based on the cutoff $c^{**}(\theta)$, one can then construct the investment $a^{**}(\theta)$ and payment rule $y^{**}(\theta, c)$ with $\pi(\underline{\theta}, \underline{\theta}) = 0$, according to conditions (ii)–(iv) in Lemma 2. Then, by construction, the cutoff mechanism $\{c^{**}(\theta), a^{**}(\theta), y^{**}(\theta, c)\}$ satisfies all constraints of Problem (O-C), so that it is feasible in Problem (O-C).

Recall the definition of $\tilde{\eta}(K_\varepsilon)$. It implies that $\varphi(\hat{c}; \theta)$ is strictly decreasing in $\hat{c} \in [c_B(\theta), c_B(\theta) + \tilde{\eta}(K_\varepsilon)]$, for all $\theta \in K_\varepsilon$. By construction, for any $\theta \in K_\varepsilon$, whenever $c^*(\theta) < c_B(\theta) + \eta_\varepsilon$, $c^{**}(\theta) \in (c^*(\theta), c_B(\theta) + \tilde{\eta}(K_\varepsilon)]$. When $c^*(\theta) < c_B(\theta) + \eta_\varepsilon$, since $(c^*(\theta), c_B(\theta) + \tilde{\eta}(K_\varepsilon)] \subseteq [c_B(\theta), c_B(\theta) + \tilde{\eta}(K_\varepsilon)]$, $\varphi(c^{**}(\theta); \theta) < \varphi(c^*(\theta); \theta)$.

Thus, the construction of $c^{**}(\theta)$ implies that for any $\theta \in K_\varepsilon$, whenever $c^*(\theta) \geq c_B(\theta) + \eta_\varepsilon$—i.e., $c^{**}(\theta) = c^*(\theta)$, $\varphi(c^{**}(\theta); \theta) = \varphi(c^*(\theta); \theta)$; whenever $c^*(\theta) < c_B(\theta) + \eta_\varepsilon$, $\varphi(c^{**}(\theta); \theta) < \varphi(c^*(\theta); \theta)$. Notice that the objective function in Problem (O-C) can be expressed as $\int_{\underline{\theta}}^{\overline{\theta}} \varphi(c(\theta); \theta) g(\theta) d\theta$. Therefore, $\{c^{**}(\theta), a^{**}(\theta), y^{**}(\theta, c)\}$ is at least as good as $\{c^*(\theta), a^*(\theta), y^*(\theta, c)\}$ pointwisely. The remaining question is whether $\varphi(c^{**}(\theta); \theta) < \varphi(c^*(\theta); \theta)$ on a positive measure subset of $[\underline{\theta}, \overline{\theta}]$.

This is indeed the case. Notice that from (32), $c^*(\theta) < c_B(\theta) + \eta_\varepsilon$ for any $\theta \in [\theta_1 - \delta_1, \theta_1]$. Thus,

$$\int_{\underline{\theta}}^{\overline{\theta}} \varphi(c^{**}(\theta); \theta) g(\theta) d\theta - \int_{\underline{\theta}}^{\overline{\theta}} \varphi(c^{**}(\theta); \theta) g(\theta) d\theta \leq \int_{\theta_1 - \delta_1}^{\theta_1} \left[\varphi(c^{**}(\theta); \theta) - \varphi(c^*(\theta); \theta)\right] g(\theta) d\theta < 0,$$

where the strict inequality follows from the fact that $\varphi(c^{**}(\theta);\theta) - \varphi(c^*(\theta);\theta) < 0$ for any $\theta \in [\theta_1 - \delta_1, \theta_1]$. However, this contradicts the optimality of $\{c^*(\theta), \alpha^*(\theta), y^*(\theta, c)\}$.

Therefore, we must have $c^*(\theta_1) > c_B(\theta_1)$, $\forall\ \theta_1 \in K_\varepsilon \backslash \{\underline{\theta}\}$.

Step 4: We have proved that $c^*(\theta) \in (c_B(\theta), c_0]$ for any $\theta \in (\underline{\theta}, \theta_\varepsilon]$ in step 3. Recall that $\varepsilon > 0$ is arbitrary and that $\theta_\varepsilon \to \overline{\theta}$ as $\varepsilon \to 0$. Therefore, letting ε approach 0 establishes the result that $c^*(\theta) \in (c_B(\theta), c_0]$ for any $\theta \in (\underline{\theta}, \overline{\theta})$. This completes the proof. $\square$

Proof of Proposition 3: Under Assumption 3, for any θ, $\tilde{c}(\theta) < c_B(\theta)$. Thus, this is Case 1 in the proof of Lemma A4. There, in (35), when $\hat{c} > \tilde{c}(\theta)$, we have shown that

$$\frac{d\varphi(\hat{c};\theta)}{d\hat{c}} = (\hat{c} - c_0)\left[h(\hat{c}, k(\hat{c})) + \frac{d\alpha(\hat{c})}{d\hat{c}} H_2(c, k(\hat{c}))\right] + \frac{1 - G(\theta)}{g(\theta)} \frac{d\alpha(\hat{c})}{d\hat{c}} C''(\alpha(\hat{c})),$$

where $\varphi(\hat{c};\theta)$ is defined in the proof of Proposition 2. With linear cost, $C''(\cdot) = 0$, so

$$\frac{d\varphi(\hat{c};\theta)}{d\hat{c}} = (\hat{c} - c_0)\left[h(\hat{c}, k(\hat{c})) + \frac{d\alpha(\hat{c})}{d\hat{c}} H_2(c, k(\hat{c}))\right],$$

which implies that when $\hat{c} > \tilde{c}(\theta)$, $\frac{d\varphi}{d\hat{c}}$ has the same sign as $\hat{c} - c_0$, because $\frac{d\alpha(\hat{c})}{d\hat{c}} > 0$, as mentioned in the proof of Lemma A4. Therefore, $\varphi(\hat{c};\theta)$ is strictly decreasing in $\hat{c}$ when $\hat{c} \in (\tilde{c}(\theta), c_0)$ and is strictly increasing in $\hat{c}$ when $\hat{c} \in [c_0, \overline{c}]$. Recall that we have also shown in Case 1 in the proof of Lemma A4 that $\varphi(\hat{c};\theta)$ is strictly decreasing in $\hat{c}$ when $\hat{c} \in [\underline{c}, \tilde{c}(\theta)]$. Thus, $\varphi(\hat{c};\theta)$ achieves its unique minimum at $\hat{c} = c_0$, for all $\theta \in [\underline{\theta}, \overline{\theta}]$.

Therefore, for Problem (O-C-R-R) defined in the proof of Proposition 2, the optimal solution is $c^*(\theta) = c_0$, for all $\theta \in [\underline{\theta}, \overline{\theta}]$. Since $c^*(\theta) = c_0$ for all $\theta \in [\underline{\theta}, \overline{\theta}]$ satisfies the monotonicity constraint (condition (i)) in Lemma 2, based on $c^*(\theta)$, one can construct investment $a^*(\theta)$ and payment rule $y^*(\theta, c)$ with $\pi(\underline{\theta}, \underline{\theta}) = 0$, following conditions (ii)–(iv) in Lemma 2. Then, by construction, the cutoff mechanism $\{c^*(\theta), a^*(\theta), y^*(\theta, c)\}$ satisfies all constraints of Problem (O-C), so that it is feasible in Problem (O-C). Notice that the objective function in Problem (O-C) can be expressed as $\int_{\underline{\theta}}^{\overline{\theta}} \varphi(c(\theta);\theta) g(\theta) d\theta$. Therefore, $\{c^*(\theta), a^*(\theta), y^*(\theta, c)\}$ must the optimal solution of Problem (O-C). Note that from (iv) of Lemma 2, $\alpha^*(\theta) = \alpha^{FB}(\theta) = \theta^* - \theta$. $\square$

6.2 Appendix B

Proof of Lemma A1: Fix the report $\hat{\theta}$. Recall from footnote 14 that for any θ, $\alpha(\hat{\theta},\theta)$ is unique. Therefore, $\alpha(\hat{\theta},\cdot)$ is indeed a function. From equation (19), since $H_{22}(c,z) < 0$ when $c \in (\underline{c},\overline{c})$ and $C''(\cdot) \geq 0$, it is easy to see that $\alpha(\hat{\theta},\theta)$ is decreasing in θ and is strictly decreasing in θ when $\alpha(\hat{\theta},\theta) > 0$. Therefore, for any $\hat{\theta}$, there exists a unique $\delta(\hat{\theta}) \in [\underline{\theta},\overline{\theta}]$ such that $\alpha(\hat{\theta},\theta) > 0$ if and only if $\theta < \delta(\hat{\theta})$; and $\alpha(\hat{\theta},\theta) = 0$ if and only if $\theta \geq \delta(\hat{\theta})$ (the situation that $\delta(\hat{\theta}) \in [\underline{\theta},\overline{\theta}]$ does not exist will be discussed later in the proof). In other words, $\delta(\hat{\theta})$ satisfies

$$-C'(0) + \int_{\underline{c}}^{\overline{c}} p(\hat{\theta},c) H_2(c,\delta(\hat{\theta}))dc = 0. \tag{33}$$

When $\theta < \delta(\hat{\theta})$, (19) holds with equality and $\alpha(\hat{\theta},\theta)$ is strictly decreasing in θ, so the implicit function theorem implies that the function $\alpha(\hat{\theta},\theta)$ is a differentiable function of θ when $\theta < \delta(\hat{\theta})$.[15] When $\theta > \delta(\hat{\theta})$, $\alpha(\hat{\theta},\theta) = 0$ so that it is trivially differentiable in θ. As a result, the function $\alpha(\hat{\theta},\theta)$ is differentiable in $\theta \in [\underline{\theta},\overline{\theta}]$ except when $\theta = \delta(\hat{\theta})$.

Note that it is possible that $\delta(\hat{\theta}) \in [\underline{\theta},\overline{\theta}]$ does not exist, which means that either $\alpha(\hat{\theta},\theta) > 0$ for all $\theta \in [\underline{\theta},\overline{\theta}]$ or $\alpha(\hat{\theta},\theta) = 0$ for all $\theta \in [\underline{\theta},\overline{\theta}]$. In this case, $\alpha(\hat{\theta},\theta)$ is differentiable in θ everywhere in $[\underline{\theta},\overline{\theta}]$. In fact, if $\alpha(\hat{\theta},\theta) > 0$ for all $\theta \in [\underline{\theta},\overline{\theta}]$, (19) holds with equality, so arguments similar to those in the previous paragraph imply that $\alpha(\hat{\theta},\theta)$ is a differentiable function of θ. If $\alpha(\hat{\theta},\theta) = 0$ for all $\theta \in [\underline{\theta},\overline{\theta}]$, $\alpha(\hat{\theta},\theta)$ is obviously differentiable in θ.

Therefore, $\alpha(\hat{\theta},\cdot)$ is differentiable everywhere except possibly at one point. The continuity of $\alpha(\hat{\theta},\cdot)$ is obvious. $\square$

Proof of Lemma A2: If $\delta(\hat{\theta}) \in [\underline{\theta},\overline{\theta}]$ does not exist, then either $\alpha(\hat{\theta},\theta) = 0$ for all $\theta \in [\underline{\theta},\overline{\theta}]$ or $\alpha(\hat{\theta},\theta) > 0$ for all $\theta \in [\underline{\theta},\overline{\theta}]$. For the former case,

$$\pi(\hat{\theta},\theta) = \tilde{\pi}(\hat{\theta},\overline{c},\overline{c}) + \int_{\underline{c}}^{\overline{c}} p(\hat{\theta},c) H(c,\theta)dc,$$

which is differentiable in θ over $[\underline{\theta},\overline{\theta}]$ by the Lebesgue's dominated convergence theorem. Moreover,

[15] For any given $\theta' \in [\underline{\theta},\delta(\hat{\theta}))$, the equation $-C'(\alpha) + \int_{\underline{c}}^{\overline{c}} p(\hat{\theta},c)H_2(c,\theta+\alpha)dc = 0$ defines α as a differentiable function of θ in some neighborhood of θ' by the implicit function theorem. However, by footnote 14, such solution α is unique. Therefore, this implicit function must coincide with $\alpha(\hat{\theta},\cdot)$ for any $\theta' \in [\underline{\theta},\delta(\hat{\theta}))$. The differentiability of $\alpha(\hat{\theta},\theta)$ in $\theta \in [\underline{\theta},\delta(\hat{\theta}))$ then follows.

its derivative is

$$\frac{\partial \pi(\hat{\theta},\theta)}{\partial \theta} = \int_{\underline{c}}^{\overline{c}} p(\hat{\theta},c)H_2(c,\theta)dc = \int_{\underline{c}}^{\overline{c}} p(\hat{\theta},c)H_2(c,\theta+\alpha(\hat{\theta},\theta))dc,$$

which is obviously continuous in θ over $[\underline{\theta},\overline{\theta}]$ (as $\alpha(\hat{\theta},\cdot)$ is continuous). For the latter case that $\alpha(\hat{\theta},\theta) > 0$ for all $\theta \in [\underline{\theta},\overline{\theta}]$, the proof for the case when $\delta(\hat{\theta}) \in [\underline{\theta},\overline{\theta}]$ exists, as we state below, obviously applies to it.

Now suppose $\delta(\hat{\theta}) \in [\underline{\theta},\overline{\theta}]$ exists. We first show that $\pi(\hat{\theta},\theta)$ is differentiable in $\theta \in [\underline{\theta},\overline{\theta}]$. By Lemma A1, $\alpha(\hat{\theta},\cdot)$ is continuous in $[\underline{\theta},\overline{\theta}]$ and differentiable everywhere except at the point $\delta(\hat{\theta})$. Therefore, for any $\hat{\theta}$, by the Lebesgue's dominated convergence theorem,[16] $\pi(\hat{\theta},\theta)$ is differentiable when $\theta \neq \delta(\hat{\theta})$. Now we only need to show that it is also differentiable at $\theta = \delta(\hat{\theta})$. Notice that when $\theta \geq \delta(\hat{\theta})$, $\alpha(\hat{\theta},\theta) = 0$ so that $\pi(\hat{\theta},\theta) = \tilde{\pi}(\hat{\theta},\overline{c},\overline{c}) + \int_{\underline{c}}^{\overline{c}} p(\hat{\theta},c)H(c,\theta)dc$. Therefore, by the dominated convergence theorem, the right derivative

$$\lim_{\theta\to\delta(\hat{\theta})^+} \frac{\pi(\hat{\theta},\theta)-\pi(\hat{\theta},\delta(\hat{\theta}))}{\theta-\delta(\hat{\theta})} = \lim_{\theta\to\delta(\hat{\theta})^+} \frac{\int_{\underline{c}}^{\overline{c}} p(\hat{\theta},c)[H(c,\theta)-H(c,\delta(\hat{\theta}))]dc}{\theta-\delta(\hat{\theta})} = \int_{\underline{c}}^{\overline{c}} p(\hat{\theta},c)H_2(c,\delta(\hat{\theta}))dc.$$

When $\theta < \delta(\hat{\theta})$,

$$\pi(\hat{\theta},\theta) = -C(\alpha(\hat{\theta},\theta)) + \tilde{\pi}(\hat{\theta},\overline{c},\overline{c}) + \int_{\underline{c}}^{\overline{c}} p(\hat{\theta},c)H(c,\theta+\alpha(\hat{\theta},\theta))dc.$$

[16] Using the implicit function theorem, it is easy to see that the derivative of $\alpha(\hat{\theta},\cdot)$ is bounded.

The left derivative, again by the dominated convergence theorem, is

$$
\begin{aligned}
& \lim_{\theta \to \delta(\hat{\theta})^-} \frac{\pi(\hat{\theta},\theta) - \pi(\hat{\theta},\delta(\hat{\theta}))}{\theta - \delta(\hat{\theta})} \\
= & \lim_{\theta \to \delta(\hat{\theta})^-} \frac{-\left[C(\alpha(\hat{\theta},\theta)) - C(\alpha(\hat{\theta},\delta(\hat{\theta})))\right] + \int_{\underline{c}}^{\overline{c}} p(\hat{\theta},c) \left[\begin{array}{c} H(c,\theta+\alpha(\hat{\theta},\theta)) \\ -H(c,\delta(\hat{\theta})+\alpha(\hat{\theta},\delta(\hat{\theta}))) \end{array}\right] dc}{\theta - \delta(\hat{\theta})} \\
= & -C'(0)\frac{\partial \alpha(\hat{\theta},\delta(\hat{\theta})^-)}{\partial \theta} + \int_{\underline{c}}^{\overline{c}} p(\hat{\theta},c) \lim_{\theta \to \delta(\hat{\theta})^-} \frac{H(c,\theta+\alpha(\hat{\theta},\theta)) - H(c,\delta(\hat{\theta})+\alpha(\hat{\theta},\delta(\hat{\theta})))}{\theta - \delta(\hat{\theta})} dc \\
= & -C'(0)\frac{\partial \alpha(\hat{\theta},\delta(\hat{\theta})^-)}{\partial \theta} + \int_{\underline{c}}^{\overline{c}} p(\hat{\theta},c) H_2(c,\delta(\hat{\theta})) \left(1 + \frac{\partial \alpha(\hat{\theta},\delta(\hat{\theta})^-)}{\partial \theta}\right) dc \\
= & \int_{\underline{c}}^{\overline{c}} p(\hat{\theta},c) H_2(c,\delta(\hat{\theta})) dc + \frac{\partial \alpha(\hat{\theta},\delta(\hat{\theta})^-)}{\partial \theta} \underbrace{\left[-C'(0) + \int_{\underline{c}}^{\overline{c}} p(\hat{\theta},c) H_2(c,\delta(\hat{\theta})) dc\right]}_{=0 \text{ from } (33)} \\
= & \int_{\underline{c}}^{\overline{c}} p(\hat{\theta},c) H_2(c,\delta(\hat{\theta})) dc,
\end{aligned}
$$

where $\frac{\partial \alpha(\hat{\theta},\delta(\hat{\theta})^-)}{\partial \theta}$ denotes the left derivative of $\alpha(\hat{\theta},\theta)$ with respect to θ at the point $\theta = \delta(\hat{\theta})$. Thus, $\pi(\hat{\theta},\theta)$ is differentiable at $\theta = \delta(\hat{\theta})$.

Now we move on to show the other two properties. When $\theta < \delta(\hat{\theta})$, (19) holds with equality. Thus

$$
\frac{\partial \alpha(\hat{\theta},\theta)}{\partial \theta} \left(-C'(\alpha(\hat{\theta},\theta)) + \int_{\underline{c}}^{\overline{c}} p(\hat{\theta},c) H_2(c,\theta+\alpha(\hat{\theta},\theta)) dc\right) = 0. \tag{34}
$$

When $\theta > \delta(\hat{\theta})$, $\alpha(\hat{\theta},\theta) = 0$ so that $\frac{\partial \alpha(\hat{\theta},\theta)}{\partial \theta} = 0$. The above equality still holds.

Then, when $\theta \neq \delta(\hat{\theta})$,

$$
\begin{aligned}
\frac{\partial \pi(\hat{\theta},\theta)}{\partial \theta} &= -C'(\alpha(\hat{\theta},\theta))\frac{\partial \alpha(\hat{\theta},\theta)}{\partial \theta} + \int_{\underline{c}}^{\overline{c}} p(\hat{\theta},c) H_2(c,\theta+\alpha(\hat{\theta},\theta))(1 + \frac{\partial \alpha(\hat{\theta},\theta)}{\partial \theta}) dc \\
&= \int_{\underline{c}}^{\overline{c}} p(\hat{\theta},c) H_2(c,\theta+\alpha(\hat{\theta},\theta)) dc.
\end{aligned}
$$

Since $\alpha(\hat{\theta}, \theta)$ is continuous in θ, $\frac{\partial \pi(\hat{\theta},\theta)}{\partial \theta}$ is continuous when $\theta \neq \delta(\hat{\theta})$. Notice that

$$\lim_{\theta \to \delta(\hat{\theta})} \frac{\partial \pi(\hat{\theta}, \theta)}{\partial \theta} = \int_{\underline{c}}^{\overline{c}} p(\hat{\theta}, c) H_2(c, \delta(\hat{\theta})) dc,$$

which is equal to the derivative of $\pi(\hat{\theta}, \theta)$ at $\theta = \delta(\hat{\theta})$. Therefore, $\pi(\hat{\theta}, \theta)$ is continuously differentiable in $[\underline{\theta}, \overline{\theta}]$. $\square$

Proof of Lemma A3: It is easy to see that $\beta_i(t)$ is increasing in t. Therefore, there exists some t_i such that $\beta_i(t_i) = 0$ if and only if $t \leq t_i$, and $\beta_i(t_i) > 0$ if and only if $t > t_i$ (ignore the issue that $t_i \notin [\underline{c}, \overline{c}]$, as this does not affect the analysis). Similar to the proof of Lemma A1, $\beta_i(t)$ is not differentiable only when $t = t_i$. Also note that $\beta_i(t)$ is continuous. As a result, $g(t)$ is continuous everywhere and differentiable everywhere except possibly at t_1 and t_2. However, such nondifferentiability does not affect the monotonicity of $g(t)$.[17] It is easy to see that when $g(t)$ is differentiable, its derivative equals $H(t, \theta_2 + \beta_2(t)) - H(t, \theta_1 + \beta_1(t))$. If $\theta_2 + \beta_2(t) \geq \theta_1 + \beta_1(t)$, then the derivative is nonnegative, which further implies that $g(t)$ is increasing.

Thus, it suffices to prove $\theta_2 + \beta_2(t) \geq \theta_1 + \beta_1(t)$ for all $t \in [\underline{c}, \overline{c}]$. To see this, note first that $\beta_2(t) \leq \beta_1(t)$, as the function $C(\alpha) - \int_{\underline{c}}^{t} H(c, \theta + \alpha) dc$ has increasing difference in (α, θ). Thus, if $\beta_1(t) = 0$, then $\beta_1(t) = \beta_2(t) = 0$, so $\theta_2 + \beta_2(t) = \theta_2 > \theta_1 = \theta_1 + \beta_1(t)$. If $\beta_1(t) > 0$, then $C'(\beta_1(t)) = \int_{\underline{c}}^{t} H_2(c, \theta_1 + \beta_1(t)) dc$, so

$$\int_{\underline{c}}^{t} H_2(c, \theta_2 + \beta_2(t)) dc \leq C'(\beta_2(t)) \leq C'(\beta_1(t)) = \int_{\underline{c}}^{t} H_2(c, \theta_1 + \beta_1(t)) dc.$$

This further implies that $\theta_2 + \beta_2(t) \geq \theta_1 + \beta_1(t)$. The proof completes. $\square$

Proof of Lemma A4: We first prove the first statement of the lemma. Since $\alpha(\hat{c})$ is increasing in $\hat{c}$ and is strictly increasing when $\alpha(\hat{c}) > 0$, there exists a unique $\tilde{c}(\theta) \in (\underline{c}, \overline{c}]$ such that $\alpha(\hat{c}) = 0$ if and only if $\hat{c} \in [\underline{c}, \tilde{c}(\theta)]$, and $\alpha(\hat{c}) > 0$ if and only if $\hat{c} \in (\tilde{c}(\theta), \overline{c}]$ (note that $(\tilde{c}(\theta), \overline{c}]$ can be empty when $\tilde{c}(\theta) = \overline{c}$). Notice that when $\alpha(\hat{c}) = 0$,

$$c + \frac{1 - G(\theta)}{g(\theta)} \frac{H_2(c, \theta + \alpha(\hat{c}))}{h(c, \theta + \alpha(\hat{c}))} = J(c, \theta).$$

Arguments similar to those in the proof of Lemma A1 lead to that $\alpha(\hat{c})$ is differentiable in $\hat{c}$ everywhere except possibly at $\hat{c} = \tilde{c}(\theta)$ and is continuous in $\hat{c}$. This implies that $\varphi(\hat{c}; \theta)$ is

[17] In fact, it is easy to show that $g(t)$ is differentiable at t_1 and t_2, so $g(t)$ is actually differentiable everywhere.

continuous in $\hat{c}$ and is differentiable in $\hat{c}$ everywhere except possibly at $\hat{c} = \tilde{c}(\theta)$. There are two cases to consider.

Case 1: $\tilde{c}(\theta) \leq c_B(\theta)$. In this case, $\alpha(\hat{c}) = 0$ when $\hat{c} \leq \tilde{c}(\theta)$. Thus, $\varphi(\hat{c};\theta) = \int_{\underline{c}}^{\hat{c}}(J(c,\theta) - c_0)h(c,\theta)dc + c_0$, when $\hat{c} \leq \tilde{c}(\theta)$. By the definition of $c_B(\theta)$ in (2) and Assumption 1, $J(c,\theta) < c_0$ when $c < c_B(\theta)$, which implies that $\varphi(\hat{c};\theta)$ is strictly decreasing in $\hat{c}$ when $\hat{c} \in [\underline{c}, \tilde{c}(\theta)]$.

When $\hat{c} > \tilde{c}(\theta)$, (29) is binding. Then (to simplify notation, let $k(\hat{c}) = \theta + \alpha(\hat{c})$):

$$\begin{aligned}
\varphi(\hat{c};\theta) &= C(\alpha(\hat{c})) + \int_{\underline{c}}^{\hat{c}} \frac{1-G(\theta)}{g(\theta)} H_2(c,k(\hat{c}))dc + \int_{\underline{c}}^{\hat{c}} (c - c_0)dH(c,k(\hat{c})) + c_0 \\
&= C(\alpha(\hat{c})) + \frac{1-G(\theta)}{g(\theta)} C'(\alpha(\hat{c})) + (\hat{c} - c_0)H(\hat{c},k(\hat{c})) - \int_{\underline{c}}^{\hat{c}} H(c,k(\hat{c}))dc + c_0.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\frac{d\varphi}{d\hat{c}} &= \frac{d\alpha(\hat{c})}{d\hat{c}}[C'(\alpha(\hat{c})) + \frac{1-G(\theta)}{g(\theta)}C''(\alpha(\hat{c}))] + H(\hat{c},k(\hat{c})) + (\hat{c} - c_0)\left[h(\hat{c},k(\hat{c})) + \frac{d\alpha(\hat{c})}{d\hat{c}}H_2(c,k(\hat{c}))\right] \\
&\quad -H(\hat{c},k(\hat{c})) - \frac{d\alpha(\hat{c})}{d\hat{c}}\int_{\underline{c}}^{\hat{c}} H_2(c,k(\hat{c}))dc \\
&= (\hat{c} - c_0)\left[h(\hat{c},k(\hat{c})) + \frac{d\alpha(\hat{c})}{d\hat{c}}H_2(c,k(\hat{c}))\right] + \frac{1-G(\theta)}{g(\theta)}\frac{d\alpha(\hat{c})}{d\hat{c}}C''(\alpha(\hat{c})). \qquad (35)
\end{aligned}$$

Note that $\frac{d\alpha(\hat{c})}{d\hat{c}} > 0$ when $\hat{c} > \tilde{c}(\theta)$. Thus, $\frac{d\varphi}{d\hat{c}} \geq 0$ when $\hat{c} \geq c_0$, with strict inequality when $\hat{c} > c_0$. This means that $\varphi(\hat{c};\theta)$ is strictly increasing when $\hat{c} \in [c_0, \overline{c}]$.

Now consider the case when $\hat{c} \in (\tilde{c}(\theta), c_B(\theta))$. Since (29) is binding when $\hat{c} > \tilde{c}(\theta)$,

$$C'(\alpha(\hat{c})) = \int_{\underline{c}}^{\hat{c}} H_2(c,k(\hat{c}))dc, \ \ \hat{c} > \tilde{c}(\theta).$$

Taking derivative with respect to $\hat{c}$ on both sides of the equation leads to

$$\frac{d\alpha(\hat{c})}{d\hat{c}}C''(\alpha(\hat{c})) = H_2(\hat{c},k(\hat{c})) + \frac{d\alpha(\hat{c})}{d\hat{c}}\int_{\underline{c}}^{\hat{c}} H_{22}(c,k(\hat{c}))dc. \qquad (36)$$

Therefore, the expression of $\frac{d\varphi}{d\hat{c}}$ in (35) can be written as

$$\begin{aligned}
\frac{d\varphi}{d\hat{c}} &= (\hat{c}-c_0)\begin{bmatrix} h(\hat{c},k(\hat{c})) \\ +\frac{d\alpha(\hat{c})}{d\hat{c}}H_2(c,k(\hat{c})) \end{bmatrix} + \frac{1-G(\theta)}{g(\theta)}\begin{bmatrix} H_2(\hat{c},k(\hat{c})) \\ +\frac{d\alpha(\hat{c})}{d\hat{c}}\int_{\underline{c}}^{\hat{c}} H_{22}(c,k(\hat{c}))dc \end{bmatrix} \\
&= \underbrace{h(\hat{c},k(\hat{c}))\begin{bmatrix} \hat{c}-c_0 \\ +\frac{1-G(\theta)}{g(\theta)}\frac{H_2(\hat{c},k(\hat{c}))}{h(\hat{c},k(\hat{c}))} \end{bmatrix}}_{A} + \underbrace{\frac{d\alpha(\hat{c})}{d\hat{c}}\begin{bmatrix} (\hat{c}-c_0)H_2(c,k(\hat{c})) \\ +\frac{1-G(\theta)}{g(\theta)}\int_{\underline{c}}^{\hat{c}} H_{22}(c,k(\hat{c}))dc \end{bmatrix}}_{B}. \qquad (37)
\end{aligned}$$

We claim that $\frac{d\varphi}{d\hat{c}} < 0$ when $\hat{c} \in (\tilde{c}(\theta), c_B(\theta))$. To see this, note that when $\hat{c} \in (\tilde{c}(\theta), c_B(\theta))$, $B < 0$, because $\frac{d\alpha(\hat{c})}{d\hat{c}} > 0$, $\hat{c} - c_0 < 0$, $H_2(c,k(\hat{c})) > 0$, and $H_{22}(c,k(\hat{c})) < 0$ for any $c \in (\underline{c}, \hat{c}]$. Thus, it suffices to show that $A < 0$. To this end, notice that A has the same sign as

$$\hat{c} - c_0 + \frac{1-G(\theta)}{g(\theta)}\frac{H_2(\hat{c},k(\hat{c}))}{h(\hat{c},k(\hat{c}))} = \hat{c} + \frac{1-G(\theta)}{g(\theta)}\frac{H_2(\hat{c},\theta+\alpha(\hat{c}))}{h(\hat{c},\theta+\alpha(\hat{c}))} - c_0. \qquad (38)$$

Our goal is to show that the RHS of (38) is negative. In fact, when $\hat{c} \in (\tilde{c}(\theta), c_B(\theta))$, $J(\hat{c},\theta) < c_0$. Furthermore, Assumption 2 implies that for any $\hat{c} \in (\tilde{c}(\theta), c_B(\theta))$,

$$J(\hat{c},\theta) = \hat{c} + \frac{1-G(\theta)}{g(\theta)}\frac{H_2(\hat{c},\theta)}{h(\hat{c},\theta)} \geq \hat{c} + \frac{1-G(\theta)}{g(\theta)}\frac{H_2(\hat{c},\theta+\alpha(\hat{c}))}{h(\hat{c},\theta+\alpha(\hat{c}))}.$$

Therefore, when $\hat{c} \in (\tilde{c}(\theta), c_B(\theta))$,

$$\hat{c} + \frac{1-G(\theta)}{g(\theta)}\frac{H_2(\hat{c},\theta+\alpha(\hat{c}))}{h(\hat{c},\theta+\alpha(\hat{c}))} - c_0 \leq \hat{c} + \frac{1-G(\theta)}{g(\theta)}\frac{H_2(\hat{c},\theta)}{h(\hat{c},\theta)} - c_0 = J(\hat{c},\theta) - c_0 < 0.$$

Thus, we have established that $\varphi(\hat{c};\theta)$ is strictly decreasing in $\hat{c}$ when $\hat{c} \in (\tilde{c}(\theta), c_B(\theta))$. Recall that we have shown above that $\varphi(\hat{c};\theta)$ is strictly decreasing in $\hat{c}$ when $\hat{c} \in [\underline{c}, \tilde{c}(\theta)]$, and that $\varphi(\hat{c};\theta)$ is strictly increasing when $\hat{c} \in [c_0, \overline{c}]$. It then follows that in Case 1, $\varphi(\hat{c};\theta) > \varphi(c_B(\theta);\theta)$ when $\hat{c} < c_B(\theta)$, and $\varphi(\hat{c};\theta) > \varphi(c_0;\theta)$ when $\hat{c} > c_0$.

Case 2: $\tilde{c}(\theta) > c_B(\theta)$. In this case, when $\hat{c} \leq \tilde{c}(\theta)$, $\alpha(\hat{c}) = 0$. Then $\varphi(\hat{c};\theta) = \int_{\underline{c}}^{\hat{c}}(J(c,\theta) - c_0)h(c,\theta)dc + c_0$ when $\hat{c} \leq \tilde{c}(\theta)$. Thus, by the definition of $c_B(\theta)$ in (2) and Assumption 1, $\varphi(\hat{c};\theta)$ is strictly decreasing when $\hat{c} \in [\underline{c}, c_B(\theta)]$, and strictly increasing when $\hat{c} \in [c_B(\theta), \tilde{c}(\theta)]$. As a result, $\varphi(\hat{c};\theta) > \varphi(c_B(\theta);\theta)$ when $\hat{c} < c_B(\theta)$.

There are two subcases when considering $\hat{c} > c_0$.

Case 2.1: $\tilde{c}(\theta) \leq c_0$. In this case, arguments similar to those in Case 1 imply that $\varphi(\hat{c};\theta)$ is strictly increasing in $[c_0, \overline{c}]$. This implies that $\varphi(\hat{c};\theta) > \varphi(c_0;\theta)$ when $\hat{c} > c_0$.

Case 2.2: $\tilde{c}(\theta) > c_0$. In this case, arguments similar to those in Case 1 imply that $\varphi(\hat{c};\theta)$ is strictly increasing when $\hat{c} > \tilde{c}(\theta)$. When $\hat{c} \in [c_0, \tilde{c}(\theta)]$, $\alpha(\hat{c}) = 0$ so that $\varphi(\hat{c};\theta) = \int_{\underline{c}}^{\hat{c}}(J(c,\theta) - c_0)h(c,\theta)dc + c_0$. By the definition of $c_B(\theta)$ in (2) and Assumption 1, $\varphi(\hat{c};\theta)$ is strictly increasing when $\hat{c} \in [c_0, \tilde{c}(\theta)]$. Therefore, in this case, we still have $\varphi(\hat{c};\theta) > \varphi(c_0;\theta)$ when $\hat{c} > c_0$.

Thus, we have established that in Case 2, $\varphi(\hat{c};\theta) > \varphi(c_B(\theta);\theta)$ when $\hat{c} < c_B(\theta)$, and $\varphi(\hat{c};\theta) > \varphi(c_0;\theta)$ when $\hat{c} > c_0$. This completes the proof of the first statement of the lemma.

Now we proceed to the second statement. In fact, it suffices to show that under Assumption 3, when $\theta < \overline{\theta}$,

$$\frac{d\varphi(\hat{c};\theta)}{d\hat{c}}|_{\hat{c}=c_B(\theta)} < 0.$$

Assumption 3 implies that $\tilde{c}(\theta) < c_B(\theta)$; in other words, $\alpha(\hat{c}) > 0$ for any $\hat{c} \geq c_B(\theta)$. (We abuse the notation a bit.) Then the expression of $\frac{d\varphi(\hat{c};\theta)}{d\hat{c}}$ in (37) applies here. Note that in (37), $B < 0$ for any $\hat{c} \in (\tilde{c}(\theta), c_0)$, so $B < 0$ when $\hat{c} = c_B(\theta)$. Therefore, it suffices to show that $A \leq 0$ at $\hat{c} = c_B(\theta)$. This further boils down to showing that

$$c_B(\theta) - c_0 + \frac{1-G(\theta)}{g(\theta)} \frac{H_2(c_B(\theta), k(c_B(\theta)))}{h(c_B(\theta), k(c_B(\theta)))} \leq 0.$$

To this end, notice that the LHS of the above inequality is

$$c_B(\theta) - c_0 + \frac{1-G(\theta)}{g(\theta)} \frac{H_2(c_B(\theta), \theta + \alpha(c_B(\theta)))}{h(c_B(\theta), \theta + \alpha(c_B(\theta)))}.$$

Now Assumption 3 implies that $\alpha(c_B(\theta)) > 0$, which further implies that the above expression, under Assumption 2, satisfies

$$\begin{aligned} & c_B(\theta) - c_0 + \frac{1-G(\theta)}{g(\theta)} \frac{H_2(c_B(\theta), \theta + \alpha(c_B(\theta)))}{h(c_B(\theta), \theta + \alpha(c_B(\theta)))} \\ \leq \quad & c_B(\theta) - c_0 + \frac{1-G(\theta)}{g(\theta)} \frac{H_2(c_B(\theta), \theta)}{h(c_B(\theta), \theta)} \\ = \quad & J(c_B(\theta), \theta) - c_0 = 0. \end{aligned}$$

Remark 1. *Under Assumptions 1, 2, and 3, for type $\theta < \overline{\theta}$, $\frac{d\varphi(\hat{c};\theta)}{d\hat{c}}$ can be expressed as (37), which is a continuous function in $(\hat{c}, \theta) \in \{(\hat{c}, \theta) : c_B(\theta) \leq \hat{c} \leq c_0,\ \underline{\theta} \leq \theta < \overline{\theta}]$. Denote this function* *as* $\rho(\hat{c},\theta)$. *Denote further the smallest solution to* $\frac{d\varphi(\hat{c};\theta)}{d\hat{c}} = 0$ *in* $\hat{c} \in [c_B(\theta), c_0]$ *as* $\hat{c}^*(\theta)$.[18] *Then* $\hat{c}^*(\theta) \in (c_B(\theta), c_0]$ *and for each fixed* θ, $\varphi(\hat{c};\theta)$ *is strictly decreasing in* $\hat{c} \in [c_B(\theta), \hat{c}^*(\theta)]$.

The proof completes. $\square$

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