Shifting Supports in Dynamic Mechanism Design

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Abstract

The common-support assumption of future type distributions is standard in the dynamic mechanism design literature (e.g., Baron and Besanko [1], Courty and Li [5], Eső and Szentes [7], Krähmer and Strausz [9] and Pavan, Segal, and Toikka [15]). It is widely perceived that the assumption is innocuous, and yet no formal analysis of shifting supports has been provided in the literature. In this paper, we fill this gap in the setting of Eső and Szentes [7] and formally show that both analytical methodology and key insights can be extended. Our findings contribute to building up a firm foundation for enlarging the scope of technically tractable dynamic models.

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1 Introduction

Starting from the seminal work of Baron and Besanko [1], the literature of optimal dynamic mechanism design has grown rapidly.[1] Common supports—i.e., the support of an agent's period-$(t+1)$ type distributions are uniform across any type realization history—is typically assumed in the literature, such as Courty and Li [5], Eső and Szentes [7] and [8], Krähmer and Strausz [9], [10], and [11], Pavan, Segal, and Toikka [15], Deb and Said [6], Li and Shi [12], and Bergemann, Castro, and Weintraub [2], among many others. In two-period environments, this means that the support of the distribution of the agent's second-stage type is common across his first-stage types.

While the common-support assumption is appropriate in many applications, this is not always the case. Consider the following widely-used AR(1) process to model information dynamics: Agent

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[1] See, for example, Bergemann and Pavan [3], and Bergemann and Välimäki [4], for an excellent suvery of this literature.

i receives a private signal v_i in the first stage about his second-stage true valuation V_i for a good with $V_i = v_i + z_i$, where z_i is a shock which is independent of v_i.[2] If we assume that the supports of $H_{iv_i}(\cdot)$—the cumulative distribution function of V_i corresponding to type v_i—are common across v_i's, then the only possibility is that the support is the whole real line,[3] which may not look realistic, because, in practice, the valuation of the good is likely to be nonnegative and/or bounded above. Therefore, for this example, it is more reasonable to assume that $H_{iv_i}(\cdot)$ has a bounded support, which necessarily means that shifting supports must prevail.

It is widely perceived in the literature that the common support assumption is innocuous, and yet no formal analysis of shifting supports has been provided.[4] The goal of this paper is to fill this gap by allowing shifting supports in the model of Eső and Szentes [7]. We formally show that both the analytical methodology and key insights can be extended. Specifically, the revenue-maximizing mechanism takes the same form as that in Courty and Li [5] and Eső and Szentes [7]; the orthogonalized future private information does not yield any information rents for the agent; and the principal can achieve the same revenue as she could in the environment in which all agents' orthogonalized second-stage information is public. Our findings thus help to provide a firm foundation for enlarging the scope of technically tractable dynamic models.

While the main procedure for establishing our results resembles that of Eső and Szentes [7], three notable issues arise when allowing shifting supports: 1) pinning down the second stage optimal strategy following a lie in the first stage; 2) characterizing the first stage off-equilibrium payoff; and 3) establishing the incentive compatibility of the candidate optimal mechanism.

To elaborate, in Eső and Szentes [7], the agent will fully correct his first-stage lie by reporting a second-stage (orthogonalized) signal such that combining with the lie it leads to the same *ex post* valuation (i.e., second-stage valuation). What makes the full lie-correction feasible is the common-support assumption for the *ex post* valuation. With shifting supports, clearly full lie-correction is not always feasible. We show that the agents' second stage optimal strategy is to correct his lie as much as possible, because of the first-order stochastic dominance assumption in Eső and Szentes [7]. Due to its non-differentiability nature, this new strategy, in turn, generates complications in characterizing the off-equilibrium payoff, which further impacts the way to establish the incentive compatibility of a proposed mechanism. Nevertheless, we are able to establish a tractable form for the off-equilibrium payoff, which resembles that in Eső and Szentes [7], but with a different lie-correcting strategy.

The rest of the paper is organized as follows. Section 2 sets up the model. The main analysis is presented in Section 3. Section 4 concludes and Section 5 collects technical proofs.

[2] More generally, this applies to $V_i = \rho v_i + z_i$, where $\rho > 0$ is a constant.

[3] To see this, notice that if the infimum (supremum) of the common support of $H_{iv_i}(\cdot)$ is finite, then this must imply that the supports of v_i and z_i are bounded below (above). However, this can never lead to the assumption that the supports of $H_{iv_i}(\cdot)$ are the same for all v_i's.

[4] The common support assumption together with the regularity conditions usually adopted in the literature often means that the support covers the whole real line.

2 The Model

We consider the following two-stage mechanism design problem built on Eső and Szentes [7]. There are n risk-neutral buyers for a single individual good sold by a risk-neutral seller. The seller's valuation for the good is normalized as 0 and her goal is to maximize her revenue. A buyer's payoff is his valuation for the good (if he obtains it) less his payment. At the first stage, each buyer does not know exactly his valuation for the good, which will be realized at the second stage; however, each buyer has a private signal (first-stage type) determining his true valuation. Specifically, buyer i's true valuation V_i, realized in the second stage, is a random draw from the cumulative distribution $H_{iv_i}(\cdot)$, where v_i is his first-stage private type.

Our paper departs from Eső and Szentes [7] by allowing "shifting" supports: The CDF $H_{iv_i}(\cdot)$ has a positive density $h_{iv_i}(\cdot)$ everywhere over the support $[\underline{V}_{iv_i}, \overline{V}_{iv_i}]$. Assume also that $H_{iv_i}(V_i)$ is twice continuously differentiable in v_i and V_i, $\forall i$. Buyer i's first-stage type v_i follows a CDF $F_i(\cdot)$ with strictly positive density $f_i(\cdot)$ over the support $[\underline{v}_i, \overline{v}_i]$. $F_i(\cdot)$ and $H_{iv_i}(\cdot)$ are public information, $\forall v_i, \forall i$. Buyer i's first-stage and second-stage types v_i and V_i need not be independent. However, (v_i, V_i) is assumed to be independent across i.

The timing of the game is: At time 0, Nature draws buyer i's type v_i according to $F_i(\cdot)$ and v_i is buyer i's private information, for any i. At time 1, the seller announces the mechanism, which will be executed in the second stage; every buyer decides whether to participate in the mechanism (auction); if a buyer quits, he obtains his reservation payoff which is normalized as 0. At time 2, buyer i's private true valuation for the good, V_i, is randomly drawn from $H_{iv_i}(\cdot)$, for all i; and then the mechanism is implemented.

Similar to Eső and Szentes [7], we perform the following orthogonalization. Given v_i and V_i, construct s_i as $s_i = H_{iv_i}(V_i)$, for any $V_i \in [\underline{V}_{iv_i}, \overline{V}_{iv_i}]$. It is easy to see that s_i follows the uniform distribution $U[0, 1]$ and thus is independent of v_i. Denote the CDF of s_i as $G_i(\cdot)$ and its corresponding density function as $g_i(\cdot)$. As is shown in Eső and Szentes [7], s_i and V_i contain the same information and buyer i can recover his valuation V_i through $u_i(v_i, s_i) = H_{iv_i}^{-1}(s_i)$. Thus, treating buyer i's second-stage type as s_i is equivalent to the original information environment. In what follows, we will focus on (v_i, s_i) and call s_i as the second-stage *signal*.

We make the following four assumptions[5] as in Eső and Szentes [7].

Assumption 1 $\frac{1-F_i(v_i)}{f_i(v_i)}$ *is weakly decreasing in* v_i*,* $\forall i$*.*

Assumption 2 $\frac{\partial H_{iv_i}(V_i)}{\partial v_i} < 0$*,* $\forall V_i \in (\underline{V}_{iv_i}, \overline{V}_{iv_i})$*,* $\forall i$*.*

Assumption 3 $\frac{\partial H_{iv_i}(V_i)}{\partial v_i} / h_{iv_i}(V_i)$ *is increasing in* V_i*,* $\forall i$*.*

[5] As shown in Eső and Szentes [7], Assumption 3 is equivalent to $u_{i12} \leq 0$ and Assumption 4 is equivalent to $u_{i11}/u_{i1} \leq u_{i12}/u_{i2}$. (For a two-variable function, we use subscripts 1 and 2 to represent the partial derivative with respect to the first and second argument, respectively. Notations for partial derivatives of functions with three variables are similar.)

Assumption 4 $\frac{\partial H_{iv_i}(V_i)}{\partial v_i}/h_{iv_i}(V_i)$ *is increasing in* v_i, $\forall i$.

Remark: Assumption 2 implies that for any v_i and $\hat{v}_i$ with $\hat{v}_i < v_i$, H_{iv_i} first-order stochastic dominates $H_{i\hat{v}_i}$; thus, in particular, $\overline{V}_{iv_i} \geq \overline{V}_{i\hat{v}_i}$ and $\underline{V}_{iv_i} \geq \underline{V}_{i\hat{v}_i}$. This implies that in general the support of H_{iv_i} changes when v_i changes. In Eső and Szentes [7], the supports of H_{iv_i} are common for all v_i's.

3 Analysis of Shifting Supports

3.1 Benchmark

The goal of this paper is to formally show that under shifting supports, the key insight in the dynamic mechanism design literature still holds. We first study the benchmark case in which the seller observes all buyers' second-stage signals while their first-stage types are still private. According to the revelation principle, one can restrict to direct mechanisms in which in the first stage all buyers truthfully report their first-stage types. Specifically, suppose that buyer i reports his first-stage type as v_i and his second-stage signal is s_i, then a direct mechanism can be expressed as $(\mathbf{x}(\mathbf{v},\mathbf{s}), \mathbf{t}(\mathbf{v},\mathbf{s}))$, where $\mathbf{x}(\mathbf{v},\mathbf{s}) = (x_1(\mathbf{v},\mathbf{s}),\ldots,x_n(\mathbf{v},\mathbf{s}))$ and $\mathbf{t}(\mathbf{v},\mathbf{s}) = (t_1(\mathbf{v},\mathbf{s}),\ldots,t_n(\mathbf{v},\mathbf{s}))$. Here $x_i(\mathbf{v},\mathbf{s}) = x_i(v_i,\mathbf{v}_{-i},s_i,\mathbf{s}_{-i})$ is the probability that the good is sold to buyer i, and $t_i(\mathbf{v},\mathbf{s}) = t_i(v_i,\mathbf{v}_{-i},s_i,\mathbf{s}_{-i})$ is buyer i's payment. For convenience, let $X_i(v_i,s_i)$ and $T_i(v_i,s_i)$ denote respectively the expected probability of winning and the expected payment for buyer i when his first-stage type is v_i and second-stage signal is s_i; that is

$$X_i(v_i,s_i) = \int\int x_i(v_i,\mathbf{v}_{-i},s_i,\mathbf{s}_{-i})d\mathbf{F}_{-i}(\mathbf{v}_{-i})d\mathbf{G}_{-i}(\mathbf{s}_{-i}) = \int\int x_i(v_i,\mathbf{v}_{-i},s_i,\mathbf{s}_{-i})d\mathbf{F}_{-i}(\mathbf{v}_{-i})d\mathbf{s}_{-i},$$

and

$$T_i(v_i,s_i) = \int\int t_i(v_i,\mathbf{v}_{-i},s_i,\mathbf{s}_{-i})d\mathbf{F}_{-i}(\mathbf{v}_{-i})d\mathbf{G}_{-i}(\mathbf{s}_{-i}) = \int\int t_i(v_i,\mathbf{v}_{-i},s_i,\mathbf{s}_{-i})d\mathbf{F}_{-i}(\mathbf{v}_{-i})d\mathbf{s}_{-i},$$

where $\mathbf{F}_{-i}(\mathbf{v}_{-i})$ and $\mathbf{G}_{-i}(\mathbf{s}_{-i})$ denote the CDFs of $\mathbf{v}_{-i}$ and $\mathbf{s}_{-i}$, respectively.

In the benchmark case, IC (incentive compatibility) means that all buyers will report their first-stage types truthfully. If buyer i with type v_i reports $\hat{v}_i$ in the first stage, then his expected payoff is

$$\pi_i(v_i,\hat{v}_i) = \int_0^1 [u_i(v_i,s_i)X_i(\hat{v}_i,s_i) - T_i(\hat{v}_i,s_i)]ds_i.$$

IC requires that $\pi_i(v_i,v_i) \geq \pi_i(v_i,\hat{v}_i)$, $\forall v_i,\hat{v}_i,\forall i$. Standard arguments such as envelope theorem (cf. Milgrom and Segal [13]) lead to the following result, the proof of which is omitted.[6]

[6] Since $|X_i(\hat{v}_i,s_i)| \leq 1$, using Lebesgue's dominated convergence theorem, one is easy to verify that all assumptions in Theorem 2 in Milgrom and Segal [13] are satisfied.

Proposition 1 *In the benchmark case, IC implies that*

$$\pi_i(v_i, v_i) = \pi_i(\underline{v}_i, \underline{v}_i) + \int_{\underline{v}_i}^{v_i} \int_0^1 u_{i1}(y, s_i) X_i(y, s_i) ds_i dy, \forall v_i, \forall i. \tag{1}$$

Therefore, as is standard in the literature and similar to Eső and Szentes [7], the seller's payoff, which is the sum of payments, can be expressed as

$$\int \int \sum_{i=1}^{n} [u_i(v_i, s_i) - \frac{1 - F_i(v_i)}{f_i(v_i)} u_{i1}(v_i, s_i)] x_i(v_i, \mathbf{v}_{-i}, s_i, \mathbf{s}_{-i}) d\mathbf{F}(\mathbf{v}) d\mathbf{s} - \sum_{i=1}^{n} \pi_i(\underline{v}_i, \underline{v}_i), \tag{2}$$

where $\mathbf{F}(\mathbf{v})$ denotes the CDF of $\mathbf{v} = (v_1, \ldots, v_n)$.

Denote the adjusted virtual value as $W_i(v_i, s_i) = u_i(v_i, s_i) - \frac{1-F_i(v_i)}{f_i(v_i)} u_{i1}(v_i, s_i)$. Proposition 1 immediately implies the following result.[7]

Corollary 1 *In the benchmark case, ignoring ties, the optimal allocation rule is*

$$x_i^*(v_i, \mathbf{v}_{-i}, s_i, \mathbf{s}_{-i}) = 1 \quad \text{if} \quad i = \arg\max_j \{W_j(v_j, s_j), 0\}. \tag{3}$$

The seller's optimal revenue is

$$R^* = \int \int \max_i \{W_i(v_i, s_i), 0\} d\mathbf{F}(\mathbf{v}) d\mathbf{s}. \tag{4}$$

Denote $X_i^*(v_i, s_i) = \int \int x_i^*(v_i, \mathbf{v}_{-i}, s_i, \mathbf{s}_{-i}) d\mathbf{F}_{-i}(\mathbf{v}_{-i}) d\mathbf{s}_{-i}$. Similar to Corollary 1 in Eső and Szentes [7], $X_i^*(v_i, s_i)$ has the following properties.

Corollary 2 *i) $X_i^*(v_i, s_i)$ is continuous in both v_i and s_i.*
ii) $X_i^*(v_i, s_i)$ is weakly increasing in both v_i and s_i.
iii) If $v_i > \hat{v}_i$, $s_i \leq \hat{s}_i$, and $u_i(v_i, s_i) \geq u_i(\hat{v}_i, \hat{s}_i)$, then $X_i^*(v_i, s_i) \geq X_i^*(\hat{v}_i, \hat{s}_i)$.

The proof of i) and ii) in Corollary 2 is exactly the same as that of Corollary 1 in Eső and Szentes [7]. The proof of iii) is in the appendix.

3.2 The Original Problem

In this section, we study the original problem in which the first-stage types and the second-stage signals are buyers' private information. According to the revelation principle (Myerson [14]), one can restrict to direct mechanisms denoted as (similar to the benchmark case), with a little abuse of notation, $(\mathbf{x}(\mathbf{v}, \mathbf{s}), \mathbf{t}(\mathbf{v}, \mathbf{s}))$, where $\mathbf{v} = (v_1, \ldots, v_n)$ and $\mathbf{s} = (s_1, \ldots, s_n)$ with v_i and s_i buyer i's first-stage and second-stage report, respectively.

[7] The proof is similar to that of Proposition 1 in Eső and Szentes [7], which follows by checking IC of the proposed mechanism in (3) (implied by the monotonicity of $W_i(v_i, s_i)$ in v_i).

3.2.1 Second Stage

As usual, we start from the second stage. Assuming that all buyers have truthfully reported their first-stage types. Consider buyer i with first-stage type v_i and second-stage signal s_i. His expected second-stage payoff if he reports $\hat{s}_i$ and his opponents truthfully report their second-stage signals is

$$\tilde{\pi}_i(s_i, \hat{s}_i; v_i) = u_i(v_i, s_i)X_i(v_i, \hat{s}_i) - T_i(v_i, \hat{s}_i),$$

where $X_i(v_i, \hat{s}_i)$ and $T_i(v_i, \hat{s}_i)$ denote respectively the expected probability of winning and the expected payment for buyer i when his first-stage report is v_i and his second-stage report is $\hat{s}_i$. The second-stage IC means that truth-telling is optimal, i.e., $\tilde{\pi}_i(s_i, s_i; v_i) \geq \tilde{\pi}_i(s_i, \hat{s}_i; v_i), \forall \hat{s}_i$.

Again, standard arguments such as envelope theorem lead to the following result (the proof is omitted).[8]

Lemma 1 *In the original problem, suppose that all buyers report their first-stage types truthfully. Then the second-stage IC holds if and only if the following two conditions hold: For all i,*
i)

$$\tilde{\pi}_i(s_i, s_i; v_i) = \tilde{\pi}_i(0, 0; v_i) + \int_0^{s_i} u_{i2}(v_i, z)X_i(v_i, z)dz, \forall s_i, \forall v_i; \tag{5}$$

ii) $X_i(v_i, s_i)$ *is increasing in* s_i, $\forall v_i$.

3.2.2 Optimal Deviation Following a "Lie"

Lemma 1 provides a necessary and sufficient condition for the second-stage IC, assuming truthful first stage. To characterize the first-stage IC, it is important to know buyer i's optimal deviation strategy after a "lie" in the first stage. To this end, consider buyer i with first-stage type v_i and second-stage signal s_i who reported $\hat{v}_i$ in the first stage. Eső and Szentes [7] show that the optimal deviation strategy is to correct the lie in the way that the *ex post* valuations are the same; i.e.,, buyer i will report $\hat{s}_i$ such that $u_i(\hat{v}_i, \hat{s}_i) = u_i(v_i, s_i)$. However, in general, this does not apply here, because of shifting supports—buyer i cannot find any $\hat{s}_i \in [0, 1]$ such that $u_i(\hat{v}_i, \hat{s}_i) = u_i(v_i, s_i)$, if $u_i(v_i, s_i) \notin [\underline{V}_{i\hat{v}_i}, \overline{V}_{i\hat{v}_i}]$.

We characterize the optimal deviation strategy following a lie in the first stage in the following lemma. (All proofs are relegated to the appendix.)

Lemma 2 *In an incentive-compatible two-stage mechanism, buyer i with type* v_i *who reported* $\hat{v}_i$ *in the first stage and observed signal* s_i *in the second stage will report* $\hat{s}_i = \tilde{\sigma}_i(v_i, \hat{v}_i, s_i)$ *such that:*

If $\hat{v}_i < v_i$, *then*

$$\tilde{\sigma}_i(v_i, \hat{v}_i, s_i) = \begin{cases} 1, \text{ if } u_i(v_i, s_i) > u_i(\hat{v}_i, 1), \\ \sigma_i(v_i, \hat{v}_i, s_i), \text{ otherwise.} \end{cases}$$

[8] Note that by definition, $u_i(v_i, s_i) = H^{-1}_{iv_i}(s_i)$. Thus, $u_{i2} > 0$, so that we have monotonicity ii) in Lemma 1).

If $\hat{v}_i > v_i$, then

$$\tilde{\sigma}_i(v_i, \hat{v}_i, s_i) = \begin{cases} 0, \ \textit{if} \quad u_i(v_i, s_i) < u_i(\hat{v}_i, 0), \\ \sigma_i(v_i, \hat{v}_i, s_i), \ \textit{otherwise.} \end{cases}$$

Here, $\sigma(v_i, \hat{v}_i, s_i) \in [0,1]$ is the unique signal such that $u_i(v_i, s_i) = u(\hat{v}_i, \sigma_i(v_i, \hat{v}_i, s_i))$.

Lemma 2 generalizes the observation in Lemma 4 of Eső and Szentes [7] to the shifting-supports case. It says that buyer i "tries his best" to correct his first-stage lie: If he is able to find a signal $\hat{s}_i$ in $[0,1]$ to fully mask his lie in the first stage (i.e., $u(\hat{v}_i, \hat{s}_i) = u_i(v_i, s_i)$), then he will do so by reporting $\sigma(v_i, \hat{v}_i, s_i)$; if he is unable to fully mask his lie, then he tries his best to partially mask it by "hitting the boundary"—when he under-reports ($\hat{v}_i < v_i$), first-order stochastic dominance implies that he is unable to fully correct his lie when his true valuation $u_i(v_i, s_i)$ is outside the support of $H_{i\hat{v}_i}$ (i.e., $u_i(v_i, s_i) > u_i(\hat{v}_i, 1)$), thus the best he can do is to minimize the difference so that reporting $\hat{s}_i = 1$ is optimal; similarly, when he over-reports ($\hat{v}_i > v_i$), first-order stochastic dominance implies that he is unable to fully correct his lie when his true valuation $u_i(v_i, s_i)$ is outside the support of $H_{i\hat{v}_i}$ (i.e., $u_i(v_i, s_i) < u_i(\hat{v}_i, 0)$), thus the best he can do is to minimize the difference so that reporting $\hat{s}_i = 0$ is optimal.

3.2.3 Optimal Deviation Payoff

Now we are ready to characterize buyer i's optimal deviation payoff. If buyer i with type v_i reports $\hat{v}_i$ in the first stage, his optimal expected payoff can be written as

$$\pi_i(v_i, \hat{v}_i) = \int_0^1 [u_i(v_i, s_i) X_i(\hat{v}_i, \tilde{\sigma}_i(v_i, \hat{v}_i, s_i)) - T_i(\hat{v}_i, \tilde{\sigma}_i(v_i, \hat{v}_i, s_i))] ds_i.$$

Lemma 2 enables us to characterize buyer i's optimal deviation payoff in a more tractable way than the above equation, which is crucial to establish our main result, as checking whether a mechanism is IC in the original problem highly relies on the comparison between payoffs on and off the equilibrium path. Notice that unlike in Eső and Szentes [7], $\tilde{\sigma}_i(v_i, \hat{v}_i, s_i)$ in Lemma 2 can be non-differentiable in its arguments. However, it turns out that such potential non-differentiability still leads to the following nice form of optimal deviation payoff.

Lemma 3 *In an incentive-compatible two-stage mechanism, if buyer i with type v_i reports $\hat{v}_i$ in the first stage, then his first-stage expected payoff can be expressed as:*[9]

$$\pi_i(v_i, \hat{v}_i) = \pi_i(\hat{v}_i, \hat{v}_i) + \int_0^1 \int_{\hat{v}_i}^{v_i} u_{i1}(y, s_i) X_i(\hat{v}_i, \tilde{\sigma}_i(y, \hat{v}_i, s_i)) dy ds_i.$$

The form in the above lemma resembles that in Lemma 5 in Eső and Szentes [7], but with a different optimal lie-correcting strategy. However, we would like to point out that the corner

[9] $\int_a^b$ denotes $-\int_b^a$ if $a > b$.

solution issue of $\tilde{\sigma}_i(v_i, \hat{v}_i, s_i)$ makes the analysis and derivation of Lemma 3 far from trivial and quite different from that in Eső and Szentes [7].

3.2.4 First-Stage IC

First-stage IC means that $\pi_i(v_i, v_i) \geq \pi_i(v_i, \hat{v}_i)$, $\forall v_i, \hat{v}_i, \forall i$, which further implies that

$$\pi_i(v_i, v_i) \geq \int_0^1 [u_i(v_i, s_i)X_i(\hat{v}_i, s_i) - T_i(\hat{v}_i, s_i)]ds_i,$$

where the RHS of the above inequality, denoted as $\hat{\pi}_i(v_i, \hat{v}_i)$, is the expected first-stage payoff when buyer i truthfully reports his second-stage signal even if he lied in the first stage. First-stage IC implies that $\hat{\pi}_i(v_i, v_i) = \pi_i(v_i, v_i) \geq \hat{\pi}_i(v_i, \hat{v}_i)$, $\forall v_i, \hat{v}_i, \forall i$. As explained in footnote 6, standard arguments such as envelope theorem lead to the following result.

Proposition 2 *In the original problem, the first-stage IC implies that*

$$\pi_i(v_i, v_i) = \pi_i(\underline{v}_i, \underline{v}_i) + \int_{\underline{v}_i}^{v_i} \int_0^1 u_{i1}(y, s_i)X_i(y, s_i)ds_i dy, \forall v_i, \forall i. \tag{6}$$

Therefore, the seller's revenue can be expressed as

$$\int\int \sum_{i=1}^{n} [u_i(v_i, s_i) - \frac{1 - F_i(v_i)}{f_i(v_i)} u_{i1}(v_i, s_i)] x_i(v_i, \mathbf{v}_{-i}, s_i, \mathbf{s}_{-i}) d\mathbf{F}(\mathbf{v}) d\mathbf{s} - \sum_{i=1}^{n} \pi_i(\underline{v}_i, \underline{v}_i). \tag{7}$$

Notice that (1) and (6) are exactly the same, so that the seller's revenues in the benchmark case and the original problem have the same expressions. However, this does not imply that these two problems are equivalent, as an IC mechanism in the benchmark case may not be IC in the original problem. Propositions 1 and 2 immediately imply the following observation.

Corollary 3 *Suppose that* $(\mathbf{x}(\mathbf{v}, \mathbf{s}), \mathbf{t}(\mathbf{v}, \mathbf{s}))$ *and* $(\mathbf{x}(\mathbf{v}, \mathbf{s}), \bar{\mathbf{t}}(\mathbf{v}, \mathbf{s}))$ *are IC in the original problem and the benchmark case, respectively. Then for each* i*, there exists a constant* c_i *such that*

$$T_i(v_i) - \bar{T}_i(v_i) = c_i,$$

where $T_i(v_i) = \int\int t_i(v_i, \mathbf{v}_{-i}, s_i, \mathbf{s}_{-i}) d\mathbf{F}(\mathbf{v}_{-i}) d\mathbf{s}$ *and* $\bar{T}_i(v_i) = \int\int \bar{t}_i(v_i, \mathbf{v}_{-i}, s_i, \mathbf{s}_{-i}) d\mathbf{F}(\mathbf{v}_{-i}) d\mathbf{s}$ *are the expected payment for type* v_i *under the original problem and benchmark case, respectively.*

3.2.5 Irrelevance of the Privateness of Second-Stage Signal

Recall that we have not introduced the individual rationality (IR) conditions for buyers. We refer to the following one as the IR constraints.

$$\pi_i(v_i, v_i) \geq 0, \forall v_i, \forall i. \tag{8}$$

We call $\mathbf{x}(\mathbf{v}, \mathbf{s})$ an implementable allocation rule, if there exists a payment rule $\mathbf{t}(\mathbf{v}, \mathbf{s})$ such that $(\mathbf{x}(\mathbf{v}, \mathbf{s}), \mathbf{t}(\mathbf{v}, \mathbf{s}))$ is IC in the original problem. Obviously, an IC mechanism in the original problem is also IC in the benchmark case.

We are in a position to present our first main result, which generalizes the one in Eső and Szentes [7] to the shifting-supports case and echoes the result in Eső and Szentes [8].

Proposition 3 *Suppose that* $\mathbf{x}(\mathbf{v}, \mathbf{s})$ *is implementable in the original problem. If payment rule* $\mathbf{t}(\mathbf{v}, \mathbf{s})$ *implements* $\mathbf{x}(\mathbf{v}, \mathbf{s})$ *in the original problem subject to (8), then there exists a payment rule* $\bar{\mathbf{t}}(\mathbf{v}, \mathbf{s})$ *implements* $\mathbf{x}(\mathbf{v}, \mathbf{s})$ *subject to (8) in the benchmark case. Moreover,* $T_i(v_i) = \bar{T}_i(v_i), \forall v_i, \forall i$.

The above proposition follows directly from Corollary 3 (by setting $\bar{\mathbf{t}}(\mathbf{v}, \mathbf{s}) = \mathbf{t}(\mathbf{v}, \mathbf{s})$) and the fact that any IC mechanism in the original problem is also IC in the benchmark case. An immediate implication of Proposition 3 is that for any implementable allocation rule in the original problem, the seller can implement it achieving the same revenue as if she were able to observe all buyers' orthogonalized second-stage types. In other words, all the buyers only receive information rents for their first-stage private information.

3.2.6 Implementability of the Optimum

In this subsection, we present our second main result that the revenue upper bound of the benchmark case can be achieved in the original problem, which generalizes the one in Eső and Szentes [7] to the shifting-supports case.

Notice that Proposition 3 itself does not imply that the benchmark case and the original problem are equivalent. It only states that *if* an allocation rule is implementable in the original problem, it can be done so without any revenue loss compared to the benchmark case. Recall that the revenue-maximizing allocation rule in the benchmark case is characterized in (3) in Corollary 1. Therefore, if the allocation rule $\mathbf{x}^*(\mathbf{v}, \mathbf{s})$ in (3) can be shown to be IC in the original problem, then by Proposition 3, the optimal revenue in Corollary 1 can also be achieved in the original problem. This is formally shown in the next proposition.

Proposition 4 $\mathbf{x}^*(\mathbf{v}, \mathbf{s})$ *in (3) is implementable in the original problem subject to (8). As a result, the highest revenue achievable in the original problem is also*

$$R^* = \int \int \max_i \{W_i(v_i, s_i), 0\} d\mathbf{F}(\mathbf{v}) d\mathbf{s}.$$

The critical step of the proof of the above proposition uses Lemma 3. We would like to point out that, as mentioned right before Lemma 3, the verification of IC is crucial to establish our main result Proposition 4, which highly relies on the characterization of the off-equilibrium payoff $\pi_i(v_i, \hat{v}_i)$ in Lemma 3 and its tractability in terms of comparison to the equilibrium payoff.

4 Concluding Remarks

In this paper, we study optimal mechanism design in the setting of Eső and Szentes [7] while further allowing shifting supports. Several notable issues arise, such as characterizing the optimal second stage strategy following a first stage lie, pinning down off-equilibrium payoff (Lemma 3) and establishing the incentive compatibility of the candidate mechanism. We find these issues can be overcome. As a result, the methodology and key insights in the literature can be extended to environments with shifting supports. Our findings, on one hand, justify the adoption of common support in the literature as a convenient assumption; on the other hand, help to enlarge the scope of dynamic models which are technically tractable.

5 Appendix

Proof of iii) in Corollary 2: There are two cases:

Case 1: $u_i(v_i, s_i) = u_i(\hat{v}_i, \hat{s}_i)$, then it is exactly iii) in Corollary 1 in Eső and Szentes [7], so that $X_i^*(v_i, s_i) \geq X_i^*(\hat{v}_i, \hat{s}_i)$.

Case 2: $u_i(v_i, s_i) > u_i(\hat{v}_i, \hat{s}_i)$. By Assumption 2, differentiability of $u_i(\cdot, \cdot)$, and the fact that $u_i(\hat{v}_i, s_i) \leq u_i(\hat{v}_i, \hat{s}_i)$, there exists a $\tilde{v}_i \in [\hat{v}_i, v_i)$ such that $u_i(\tilde{v}_i, s_i) = u_i(\hat{v}_i, \hat{s}_i)$. Case 1 implies that $X_i^*(\tilde{v}_i, s_i) \geq X_i^*(\hat{v}_i, \hat{s}_i)$; ii) of Corollary 2 implies that $X_i^*(v_i, s_i) \geq X_i^*(\tilde{v}_i, s_i)$. Therefore, $X_i^*(v_i, s_i) \geq X_i^*(\tilde{v}_i, s_i) \geq X_i^*(\hat{v}_i, \hat{s}_i)$. $\square$

Proof of Lemma 2: Assume that buyer i with type v_i reported $\hat{v}_i$ in the first stage and observed signal s_i in the second stage. His goal is to choose an $\hat{s}_i \in [0, 1]$ to maximize his second-stage payoff $\tilde{\pi}_i(s_i, \hat{s}_i; v_i)$, i.e.,

$$\max_{\hat{s}_i \in [0,1]} u_i(v_i, s_i) X_i(\hat{v}_i, \hat{s}_i) - T_i(\hat{v}_i, \hat{s}_i). \tag{9}$$

We focus on the case when $\hat{v}_i < v_i$. The other case $\hat{v}_i > v_i$ is similar. As for the case $\hat{v}_i < v_i$, we have two cases to consider.

Case 1: $u_i(\hat{v}_i, 1) \geq u_i(v_i, s_i)$. In this case, there exists a unique $\hat{s}_i \in [0, 1]$ such that $u_i(\hat{v}_i, \hat{s}_i) = u_i(v_i, s_i)$. Then Lemma 4 in Eső and Szentes [7] implies that reporting $\sigma_i(v_i, \hat{v}_i, s_i)$ is optimal.

Case 2: $u_i(\hat{v}_i, 1) < u_i(v_i, s_i)$. In this case, we will show that reporting $\hat{s}_i = 1$ is optimal. In fact, IC in the second stage requires that $\tilde{\pi}_i(s_i, s_i; v_i) \geq \tilde{\pi}_i(s_i, \hat{s}_i; v_i)$ for any $\hat{s}_i \in [0, 1]$, i.e.,

$$\begin{aligned} u_i(v_i, s_i) X_i(v_i, s_i) - T_i(v_i, s_i) &\geq u_i(v_i, s_i) X_i(v_i, \hat{s}_i) - T_i(v_i, \hat{s}_i) \\ &= [u_i(v_i, s_i) - u_i(v_i, \hat{s}_i)] X_i(v_i, \hat{s}_i) + u_i(v_i, \hat{s}_i) X_i(v_i, \hat{s}_i) - T_i(v_i, \hat{s}_i). \end{aligned}$$

That is equivalent to

$$\tilde{\pi}_i(s_i, s_i; v_i) - \tilde{\pi}_i(\hat{s}_i, \hat{s}_i; v_i) \geq [u_i(v_i, s_i) - u_i(v_i, \hat{s}_i)] X_i(v_i, \hat{s}_i). \tag{10}$$

Note that

$$\begin{aligned} u_i(v_i,s_i)X_i(\hat{v}_i,\hat{s}_i)-T_i(\hat{v}_i,\hat{s}_i) &= [u_i(v_i,s_i)-u_i(\hat{v}_i,\hat{s}_i)]X_i(\hat{v}_i,\hat{s}_i)+u_i(\hat{v}_i,\hat{s}_i)X_i(\hat{v}_i,\hat{s}_i)-T_i(\hat{v}_i,\hat{s}_i) \\ &= [u_i(v_i,s_i)-u_i(\hat{v}_i,\hat{s}_i)]X_i(\hat{v}_i,\hat{s}_i)+\tilde{\pi}_i(\hat{s}_i,\hat{s}_i;\hat{v}_i). \end{aligned}$$

For any $\hat{s}_i' < \hat{s}_i'' \in [0,1]$, recall from Corollary 1 that $X_i(\hat{v}_i,\hat{s}_i'') \geq X_i(\hat{v}_i,\hat{s}_i')$. Then,

$$\begin{aligned} & u_i(v_i,s_i)X_i(\hat{v}_i,\hat{s}_i'')-T_i(\hat{v}_i,\hat{s}_i'')-[u_i(v_i,s_i)X_i(\hat{v}_i,\hat{s}_i')-T_i(\hat{v}_i,\hat{s}_i')] \\ = \; & \underbrace{[u_i(v_i,s_i)-u_i(\hat{v}_i,\hat{s}_i'')]}_{>0 \text{ in case 2}}X_i(\hat{v}_i,\hat{s}_i'')+\tilde{\pi}_i(\hat{s}_i'',\hat{s}_i'';\hat{v}_i)-[u_i(v_i,s_i)-u_i(\hat{v}_i,\hat{s}_i')]X_i(\hat{v}_i,\hat{s}_i')-\tilde{\pi}_i(\hat{s}_i',\hat{s}_i';\hat{v}_i) \\ \geq \; & [u_i(v_i,s_i)-u_i(\hat{v}_i,\hat{s}_i'')]X_i(\hat{v}_i,\hat{s}_i')+\tilde{\pi}_i(\hat{s}_i'',\hat{s}_i'';\hat{v}_i)-[u_i(v_i,s_i)-u_i(\hat{v}_i,\hat{s}_i')]X_i(\hat{v}_i,\hat{s}_i')-\tilde{\pi}_i(\hat{s}_i',\hat{s}_i';\hat{v}_i) \\ = \; & -[u_i(\hat{v}_i,\hat{s}_i'')-u_i(\hat{v}_i,\hat{s}_i')]X_i(\hat{v}_i,\hat{s}_i')+\tilde{\pi}_i(\hat{s}_i'',\hat{s}_i'';\hat{v}_i)-\tilde{\pi}_i(\hat{s}_i',\hat{s}_i';\hat{v}_i) \geq 0, \end{aligned}$$

where the last inequality follows from (10). Therefore, $u_i(v_i,s_i)X_i(\hat{v}_i,\hat{s}_i)-T_i(\hat{v}_i,\hat{s}_i)$ is an increasing function in $\hat{s}_i$. This then implies that reporting $\hat{s}_i = 1$ is optimal. The proof completes. □

Proof of Lemma 3: We show this lemma for the case when $\hat{v}_i < v_i$; the other case is similar. Suppose that buyer i with type v_i who reported $\hat{v}_i$ in the first stage observes s_i in the second stage. Notice first that there exists a unique cutoff value $s_i^* \in [0,1]$ such that $u_i(v_i,s_i^*) = u_i(\hat{v}_i,1)$; if such value does not exist—i.e., $u_i(v_i,0) > u_i(\hat{v}_i,1)$—then set $s_i^* = 0$. Denote s_i^* as $\tau_i(v_i,\hat{v}_i)$.

By Lemma 2, if $s_i < \tau_i(v_i,\hat{v}_i)$, then buyer i is able to fully correct his lie by reporting $\sigma_i(v_i,\hat{v}_i,s_i)$ in the second stage; if $s_i > \tau_i(v_i,\hat{v}_i)$, then buyer i is unable to fully correct his lie in the second stage and he will report 1. Hence,

$$\begin{aligned} \pi_i(v_i,\hat{v}_i) &= \underbrace{\int_0^{\tau_i(v_i,\hat{v}_i)}[u_i(v_i,s_i)X_i(\hat{v}_i,\sigma_i(v_i,\hat{v}_i,s_i))-T_i(\hat{v}_i,\sigma_i(v_i,\hat{v}_i,s_i))]ds_i}_{U_1} \\ &\quad +\int_{\tau_i(v_i,\hat{v}_i)}^1[u_i(v_i,s_i)X_i(\hat{v}_i,1)-T_i(\hat{v}_i,1)]ds_i \\ &= U_1+\underbrace{\int_{\tau_i(v_i,\hat{v}_i)}^1[u_i(\hat{v}_i,1)X_i(\hat{v}_i,1)-T_i(\hat{v}_i,1)]ds_i}_{U_2}+\underbrace{\int_{\tau_i(v_i,\hat{v}_i)}^1[u_i(v_i,s_i)-u_i(\hat{v}_i,1)]X_i(\hat{v}_i,1)ds_i}_{U_3}. \end{aligned}$$

For U_1, buyer i is able to correct his lie such that $u_i(\hat{v}_i,\sigma_i(v_i,\hat{v}_i,s_i)) = u_i(v_i,s_i)$. Thus,

$$U_1 = \int_0^{\tau_i(v_i,\hat{v}_i)}[u_i(\hat{v}_i,\sigma_i(v_i,\hat{v}_i,s_i))X_i(\hat{v}_i,\sigma_i(v_i,\hat{v}_i,s_i))-T_i(\hat{v}_i,\sigma_i(v_i,\hat{v}_i,s_i))]ds_i.$$

It is easy to see that $\sigma_i(v_i, \hat{v}_i, s_i) > s_i$. By Lemma 1, U_1 can be written as

$$U_1 = \int_0^{\tau_i(v_i,\hat{v}_i)} [\tilde{\pi}_i(s_i, s_i; \hat{v}_i) + \int_{s_i}^{\sigma_i(v_i,\hat{v}_i,s_i)} u_{i2}(\hat{v}_i, z) X_i(\hat{v}_i, z) dz] ds_i.$$

Notice that, with fixed $\hat{v}_i$ and s_i, $\sigma_i(y, \hat{v}_i, s_i)$ is a strictly increasing function in $y \in [\hat{v}_i, v_i]$. It maps $y \in [\hat{v}_i, v_i]$ to $z \in [s_i, \sigma_i(v_i, \hat{v}_i, s_i)]$, for any $s_i \in [0, \tau_i(v_i, \hat{v}_i)]$. Thus, performing change of variables by letting $z = \sigma_i(y, \hat{v}_i, s_i)$ for U_1 leads to

$$U_1 = \int_0^{\tau_i(v_i,\hat{v}_i)} [\tilde{\pi}_i(s_i, s_i; \hat{v}_i) + \int_{\hat{v}_i}^{v_i} u_{i2}(\hat{v}_i, \sigma_i(y, \hat{v}_i, s_i)) X_i(\hat{v}_i, \sigma_i(y, \hat{v}_i, s_i)) \sigma_{i1}(y, \hat{v}_i, s_i) dy] ds_i.$$

Taking derivative with respect to y on both sides of the equality $u_i(y, s_i) = u_i(\hat{v}, \sigma_i(y, \hat{v}_i, s_i))$ leads to

$$u_{i1}(y, s_i) = u_{i2}(\hat{v}_i, \sigma_i(y, \hat{v}_i, s_i)) \sigma_{i1}(y, \hat{v}_i, s_i),$$

for any $y \in [\hat{v}_i, v_i]$. Therefore,

$$U_1 = \int_0^{\tau_i(v_i,\hat{v}_i)} [\tilde{\pi}_i(s_i, s_i; \hat{v}_i) + \int_{\hat{v}_i}^{v_i} u_{i1}(y, s_i) X_i(\hat{v}_i, \sigma_i(y, \hat{v}_i, s_i)) dy] ds_i.$$

For U_2, by Lemma 1,

$$U_2 = \int_{\tau_i(v_i,\hat{v}_i)}^1 [\tilde{\pi}_i(s_i, s_i; \hat{v}_i) + \int_{s_i}^1 u_{i2}(\hat{v}_i, z) X_i(\hat{v}_i, z) dz] ds_i.$$

Fixing $\hat{v}_i$ and $s_i \in [\tau_i(v_i, \hat{v}_i), 1]$, similarly, notice that $\sigma_i(y, \hat{v}_i, s_i)$ is a strictly increasing function in $y \in [\hat{v}_i, v_i^*]$, where v_i^* is the unique value such that $u_i(v_i^*, s_i) = u_i(\hat{v}_i, 1)$. For convenience, denote v_i^* by $\sigma_i^{-1}(1, \hat{v}_i, s_i)$, i.e., $u_i(\sigma_i^{-1}(1, \hat{v}_i, s_i), s_i) = u_i(\hat{v}_i, 1)$, for any given $\hat{v}_i$ and $s_i \in [\tau_i(v_i, \hat{v}_i), 1]$. Also note that $\sigma_i(y, \hat{v}_i, s_i)$ maps $y \in [\hat{v}_i, v_i^*]$ to $z \in [s_i, 1]$. It is easy to see that $\sigma_i^{-1}(1, \hat{v}_i, s_i)$, as a function of s_i, strictly decreases in $s_i \in [\tau_i(v_i, \hat{v}_i), 1]$. And $\sigma_i^{-1}(1, \hat{v}_i, \tau_i(v_i, \hat{v}_i)) = v_i$, $\sigma_i^{-1}(1, \hat{v}_i, 1) = \hat{v}_i$. Then $\sigma_i^{-1}(1, \hat{v}_i, s_i) \in [\hat{v}_i, v_i]$ for any $s_i \in [\tau_i(v_i, \hat{v}_i), 1]$.

Now performing change of variables by letting $z = \sigma_i(y, \hat{v}_i, s_i)$ for U_2 leads to

$$U_2 = \int_{\tau_i(v_i,\hat{v}_i)}^1 [\tilde{\pi}_i(s_i, s_i; \hat{v}_i) + \int_{\hat{v}_i}^{\sigma_i^{-1}(1,\hat{v}_i,s_i)} u_{i2}(\hat{v}_i, \sigma_i(y, \hat{v}_i, s_i)) X_i(\hat{v}_i, \sigma_i(y, \hat{v}_i, s_i)) \sigma_{i1}(y, \hat{v}_i, s_i) dy] ds_i$$

Taking derivative with respect to y on both sides of the equality $u_i(y, s_i) = u_i(\hat{v}, \sigma_i(y, \hat{v}_i, s_i))$ leads to

$$u_{i1}(y, s_i) = u_{i2}(\hat{v}_i, \sigma_i(y, \hat{v}_i, s_i)) \sigma_{i1}(y, \hat{v}_i, s_i),$$

for any $y \in [\hat{v}_i, \sigma_i^{-1}(1; \hat{v}_i, s_i)]$. Hence U_2 can be rewritten as

$$U_2 = \int_{\tau_i(v_i,\hat{v}_i)}^1 [\tilde{\pi}_i(s_i, s_i; \hat{v}_i) + \int_{\hat{v}_i}^{\sigma_i^{-1}(1,\hat{v}_i,s_i)} u_{i1}(y, s_i) X_i(\hat{v}_i, \sigma_i(y, \hat{v}_i, s_i)) dy] ds_i.$$

Then,

$$
\begin{aligned}
U_1 + U_2 &= \pi(\hat{v}_i, \hat{v}_i) + \int_0^{\tau_i(v_i,\hat{v}_i)} \int_{\hat{v}_i}^{v_i} u_{i1}(y, s_i) X_i(\hat{v}_i, \sigma_i(y, \hat{v}_i, s_i)) dy ds_i \\
&\quad + \int_{\tau_i(v_i,\hat{v}_i)}^1 \int_{\hat{v}_i}^{\sigma_i^{-1}(1,\hat{v}_i,s_i)} u_{i1}(y, s_i) X_i(\hat{v}_i, \sigma_i(y, \hat{v}_i, s_i)) dy ds_i.
\end{aligned}
$$

$U_1 + U_2$ can be further written as[10]

$$
\begin{aligned}
U_1 + U_2 &= \pi(\hat{v}_i, \hat{v}_i) + \int_0^{\tau_i(v_i,\hat{v}_i)} \int_{\hat{v}_i}^{v_i} u_{i1}(y, s_i) X_i(\hat{v}_i, \tilde{\sigma}_i(y, \hat{v}_i, s_i)) dy ds_i \\
&\quad + \int_{\tau_i(v_i,\hat{v}_i)}^1 [\int_{\hat{v}_i}^{v_i} u_{i1}(y, s_i) X(\hat{v}_i, \tilde{\sigma}_i(y, \hat{v}_i, s_i)) dy - \int_{\sigma_i^{-1}(1,\hat{v}_i,s_i)}^{v_i} u_{i1}(y, s_i) X(\hat{v}_i, \tilde{\sigma}_i(y, \hat{v}_i, s_i)) dy] ds_i \\
&= \pi(\hat{v}_i, \hat{v}_i) + \int_0^1 \int_{\hat{v}_i}^{v_i} u_{i1}(y, s_i) X_i(\hat{v}_i, \tilde{\sigma}_i(y, \hat{v}_i, s_i)) dy ds_i \\
&\quad \underbrace{- \int_{\tau_i(v_i,\hat{v}_i)}^1 \int_{\sigma_i^{-1}(1,\hat{v}_i,s_i)}^{v_i} u_{i1}(y, s_i) X(\hat{v}_i, \tilde{\sigma}_i(y, \hat{v}_i, s_i)) dy ds_i.}_{U_4}
\end{aligned}
$$

Recall that when $s_i \in [\tau_i(v_i, \hat{v}_i), 1]$, for any $y \in [\sigma_i^{-1}(1, \hat{v}_i, s_i), v_i]$, we have $\sigma_i(y, \hat{v}_i, s_i) = 1$. Therefore,

$$
\begin{aligned}
U_4 &= \int_{\tau_i(v_i,\hat{v}_i)}^1 \int_{\sigma_i^{-1}(1,\hat{v}_i,s_i)}^{v_i} u_{i1}(y, s_i) X_i(\hat{v}_i, 1) dy ds_i = \int_{\tau_i(v_i,\hat{v}_i)}^1 [u_i(v_i, s_i) - u_i(\sigma_i^{-1}(1, \hat{v}_i, s_i), s_i)] X_i(\hat{v}_i, 1) dy ds_i \\
&= \int_{\tau_i(v_i,\hat{v}_i)}^1 [u_i(v_i, s_i) - u_i(\hat{v}_i, 1)] X_i(\hat{v}_i, 1) ds_i = U_3.
\end{aligned}
$$

Thus,

$$
U_1 + U_2 + U_3 = \pi(\hat{v}_i, \hat{v}_i) + \int_0^1 \int_{\hat{v}_i}^{v_i} u_{i1}(y, s_i) X_i(\hat{v}_i, \tilde{\sigma}_i(y, \hat{v}_i, s_i)) dy ds_i.
$$

This completes the proof of Lemma 3. $\square$

Proof of Proposition 4: One only needs to show that $\mathbf{x}^*(\mathbf{v}, \mathbf{s})$ is implementable in the original problem. In fact, construct the payment rule using (5) and (6) as follows:

$$
T_i^*(v_i, s_i) = u_i(v_i, s_i) X_i^*(v_i, s_i) - \int_0^{s_i} u_{i2}(v_i, z) X_i^*(v_i, z) dz - \tilde{\pi}_i^*(0, 0; v_i),
$$

[10] Notice that when $s_i \in [0, \tau_i(v_i, \hat{v}_i)]$, $\tilde{\sigma}_i(y, \hat{v}_i, s_i) = \sigma_i(y, \hat{v}_i, s_i)$ for any $y \in [\hat{v}_i, v_i]$; when $s_i \in [\tau_i(v_i, \hat{v}_i), 1]$ and $y \in [\hat{v}_i, \sigma_i^{-1}(1, \hat{v}_i, s_i)]$, $\tilde{\sigma}_i(y, \hat{v}_i, s_i) = \sigma_i(y, \hat{v}_i, s_i)$; when $s_i \in [\tau_i(v_i, \hat{v}_i), 1]$ and $y \in [\sigma_i^{-1}(1, \hat{v}_i, s_i), v_i]$, $\tilde{\sigma}_i(y, \hat{v}_i, s_i) = 1$.

where

$$\tilde{\pi}_i^*(0,0;v_i) = \int_{\underline{v}_i}^{v_i}\int_0^1 u_{i1}(y,s_i)X_i^*(y,s_i)ds_i dy - \int_0^1\int_0^{s_i} u_{i2}(v_i,z)X_i^*(v_i,z)dzds_i.$$

One is easy to verify that under $(\mathbf{x}^*(\mathbf{v},\mathbf{s}),\mathbf{t}^*(\mathbf{v},\mathbf{s}))$,

$$\tilde{\pi}_i(s_i,s_i;v_i) = \tilde{\pi}_i^*(0,0;v_i) + \int_0^{s_i} u_{i2}(v_i,z)X_i^*(v_i,z)dz, \tag{11}$$

$$\pi_i(v_i,v_i) = \pi_i(\underline{v}_i,\underline{v}_i) + \int_{\underline{v}_i}^{v_i}\int_0^1 u_{i1}(y,s_i)X_i^*(y,s_i)ds_i dy. \tag{12}$$

Now what is left is to check that $(\mathbf{x}^*(\mathbf{v},\mathbf{s}),\mathbf{t}^*(\mathbf{v},\mathbf{s}))$ is IC in the original problem. By Corollary 2, $X_i^*(v_i,s_i)$ is increasing in s_i. Together with (11), the second-stage IC follows from Lemma 1.

Checking first-stage IC is a bit involved. To this end, suppose that buyer i with type v_i reports $\hat{v}_i < v_i$ at the first stage. The case when he reports a type higher than v_i is similar. Our goal is to show that $\pi_i(v_i,\hat{v}_i) - \pi_i(v_i,v_i) \leq 0$. In fact,

$$\pi_i(v_i,\hat{v}_i) - \pi_i(v_i,v_i) = [\pi_i(v_i,\hat{v}_i) - \pi_i(\hat{v}_i,\hat{v}_i)] - [\pi_i(v_i,v_i) - \pi_i(\hat{v}_i,\hat{v}_i)]. \tag{13}$$

Recall from Lemma 3 that

$$\pi_i(v_i,\hat{v}_i) = \pi_i(\hat{v}_i,\hat{v}_i) + \int_0^1\int_{\hat{v}_i}^{v_i} u_{i1}(y,s_i)X_i^*(\hat{v}_i,\tilde{\sigma}_i(y,\hat{v}_i,s_i))dyds_i.$$

Together with (12), the RHS of (13) can be rewritten as

$$\int_0^1\int_{\hat{v}_i}^{v_i} u_{i1}(y,s_i)X_i^*(\hat{v}_i,\tilde{\sigma}_i(y,\hat{v}_i,s_i))dyds_i - \int_0^1\int_{\hat{v}_i}^{v_i} u_{i1}(y,s_i)X_i^*(y,s_i)dyds_i. \tag{14}$$

According to footnote 10, for any $s_i \in [0,1]$ and any $y \in [\hat{v}_i, v_i]$, we have $\tilde{\sigma}_i(y,\hat{v}_i,s_i) \geq s_i$ and $u_i(y,s_i) \geq u_i(\hat{v}_i,\tilde{\sigma}_i(y,\hat{v}_i,s_i))$. iii) of Corollary 2 then implies

$$X_i^*(\hat{v}_i,\sigma_i(y,\hat{v}_i,s_i)) \leq X_i^*(y,s_i),\ \forall y \in [\hat{v}_i,v_i], \forall s_i \in [0,1],$$

which further implies that (14) is non-positive. In other words, $\pi(v_i,\hat{v}_i) - \pi(v_i,v_i) \leq 0$, so that the first-stage IC holds.

We have shown above that $(\mathbf{x}^*(\mathbf{v},\mathbf{s}),\mathbf{t}^*(\mathbf{v},\mathbf{s}))$ is IC in the original problem. Obviously, it also satisfies the IR constraint (8). This then immediately implies that the highest revenue is R^* in Proposition 4. $\square$

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