The Optimal Allocation of Prizes in Contests with Costly Entry*

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Abstract

Entry cost is ubiquitous and almost unavoidable in real life contests. In this paper, we accommodate costly entry of players and study the effort-maximizing prize allocation rule in a contest environment of all-pay auction with incomplete information as in Moldovanu and Sela [9]. With free entry, Moldovanu and Sela [9] establish the optimality of winner-take-all when effort cost function is linear or concave. Costly entry introduces a new trade-off between eliciting effort from entrants and encouraging entry of contestants. For this reason, one may expect a more lenient optimal prize allocation than winner-take-all. As equilibrium entry can be stochastic, our analysis allows prize allocation rule to be contingent on the number of entrants. Surprisingly, we find the optimal prize allocation rule always awards the whole budget to the highest effort regardless of the number of entrants when cost function is linear or concave. That is, the optimality of winner-take-all is robust to costly entry.

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1 Introduction

Contests have been widely utilized in practice (e.g., R&D competitions, sports, school admissions, internal labor markets, etc.) to incentivize agents to exert productive effort by awarding prizes. Effort maximization has thus long been a natural goal for optimal design, and not surprisingly,

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optimal design of prize allocation has been well studied among many other devices as an effective instrument to enhance the efficiency of a contest. Along this line, the seminal work of Moldovanu and Sela [9] has established the effort-maximizing rule in an all-pay auction environment with incomplete information. A celebrated result is that a winner-take-all prize allocation rule is effort-maximizing when the agents' effort cost function is linear or concave. This result well rationalizes the popularity of winner-take-all as a prize allocation rule being adopted in many contests.

Contestants often have to incur entry costs in order to participate in a contest. It is very common that agents must sacrifice other activities and thus would suffer from opportunity costs by attending a particular competition. There are many other situations such as most sports events where the players usually need to travel to a common site and stay there for the whole duration of the event, which could be quite costly. To participate in an R&D contest, the firms need to spend time and resources to collect project-related information, and set up necessary equipment before conducting the core research. For crowdsourcing contests that are typically held online, the contestants need to spend time to figure out how the platform works, and make sure they have access to proper facilities including internet connection and computers. A natural issue thus arises: given the ubiquity of entry cost, why is winner-take-all still so prevalent in reality?

The question we ask in this paper is how the existence of entry cost would affect the effort-maximizing prize allocation rule.[1] Can the winner-take-all allocation rule remains optimal when costly entry kicks in such that the organizer could forget about costly entry and act as if there were no entry cost? To address these issues, we introduce costly entry in the environment of Moldovanu and Sela [9], and study the optimal prize allocation rule.

The answers to these questions are far from obvious. A new trade-off that does not exist in the analysis of Moldovanu and Sela [9] looms large: there is a confliction between eliciting effort from entrants ex post and encouraging entry of contestants ex ante. A winner-take-all is very effective in terms of eliciting effort from entrants, however, at the same time this means it might overly discourage entry of the players. In addition, costly entry essentially leads to stochastic entry endogenously. As a result, prize allocation can be contingent on the number of entrants in general. Given that winner-take-all is clearly not a contingent allocation rule, it seems that there is lesser reason to expect it remains optimal when costly entry is introduced in. Each prize allocation rule would induce a symmetric entry threshold of players' types. A winner-take-all rule would generate (almost) the least equilibrium entry, while a more lenient allocation rule would induce more entry.[2]

[1] If the contestants' entry probability were exogenously given, then winner-take-all would be obviously optimal since the contestants' participation constraint can be ignored. Winner-take-all extracts the most surplus based on the insights of Moldovanu and Sela [9].

[2] An even less equilibrium entry can be induced by the prize allocation rule which sets the prize for the lowest rank as zero when the number of entrants is no smaller than two, while the prize when there is only one entrant is strictly smaller than the organizer's whole budget. Such kind of prize allocation rule does not extract more surplus than

The designer has to strike the optimal balance between ex post effort elicitation and ex ante entry incentive.

The trade-off is a rather complex issue, due to the complicated bidding behavior resulting from the endogenous entry and the nature of the contingent prize structure. To accommodate *endogenous and stochastic entry*, we name the situation with n entrants as *scenario* n. The organizer determines for each scenario the sum of prizes and the prize for each rank position with higher rank associated with higher prize.[3] Thus a prize allocation rule can be fully described by two vectors: the *budget vector*, which is the sum of prizes in each scenario, and the *prize allocation vector*, which specifies the prize allocation in each scenario. A change in the prize allocation rule comes along with a change in players' joint entry and bidding decisions. For this reason, the expected total effort can be too nasty to analyze.

We start with the linear cost case.[4] An important step in our analysis is to characterize all the prize allocation rules that are compatible with any given entry threshold, and identify the (unique) symmetric bidding function for any given fixed entry threshold and any corresponding compatible prize allocation rule. Clearly, it is the set of minimum prizes in all scenarios that fully determines the symmetric entry threshold type as this type for sure wins the minimum prize in each scenario. One observation about the equilibrium bidding function turns out to be quite useful: although the entrant does not observe the number of rival(s), the expected total effort can be decomposed as a weighted average of expected total effort from all scenarios,[5] and the expected total effort in each scenario, which we call *scenario expected total effort*, is identical to the expected total effort induced had the entrant(s) observed the number of rival(s). This observation reveals that the revelation policy of the number of entrants does not make a difference in optimal prize design.

Equipped with these results, we go on to characterize the optimal allocation rule while fixing the minimum prizes for all scenarios.[6] Recall that these minimum prizes jointly determine the symmetric entry threshold type. We find that the optimal prize allocation for every scenario is to set all prizes equal to the minimum prize except the highest prize, which equals the rest of the whole original budget. The reason is that the scenario n expected total effort is a linear combination of the n prizes, with the coefficients determined by the corresponding order statistics. The coefficient associated with the highest order statistics is the largest since it is most effective to induce effort

winner-take-all does ex post as the latter one already extracts the surplus to the extreme, while it strictly induces less entry than that induced by winner-take-all. Therefore, it can never be optimal. The formal proof of this argument is provided in Lemma 7.

[3] For example, for scenario n, the organizer sets the sum of prizes in that scenario which has to be no greater than her budget, and prize for each rank from the first place to the n'th place. Note that we allow the prize to be zero.

[4] In the paper, we also discuss the concave or convex cost case.

[5] The weights are the probabilities of the corresponding scenario happening.

[6] In scenario n, the minimum prize cannot be higher than $1/n$ of the initial budget due to the monotonicity constraint on prize structure.

from the highest ability entrant using a same amount of prize. Therefore, the organizer should make the first prize as large as possible. This is the principle of *cross-rank prize transfer*: transferring prize money from the lower ranked bids to the highest one induces more effort, which echoes the insight of Moldovanu and Sela [9]. Constrained by the minimum prize in that scenario, the organizer has to set the prizes for all lower ranked bids other than that for the highest bid at the level of the minimum prize.

The optimal prize allocation rule for a fixed entry threshold is then pinned down by searching through the above characterized optimal prize structure associated with each set of minimum prizes in all scenarios that induces the entry threshold. Due to the entry cost, the threshold type must obtain a strictly positive prize in some scenario(s). Recall that the expected total effort can be decomposed as a weighted average of the N scenario total efforts. A decrease of the minimum prize in a certain scenario (say, scenario n) will increase the scenario n expected total effort due to a higher first prize. However, such decrease in the minimum prize of scenario n must be compensated by an appropriate increase of the minimum prize in some other scenario (say, scenario m) to fulfil the entrant's participation constraint. Such a change would decrease the scenario m expected total effort. Thus the organizer faces the problem of how to optimally subsidize the entrant. It turns out that the higher the number of entrants is, the more effective it is to induce effort by "one weighted penny" decrease of the minimum prize in that scenario. Here, "one weighted penny" means a penny weighted by the reverse of the probability of that scenario happening from an entrant's perspective. This is rather intuitive since the higher the number of entrants n is, the higher the probability of a high ability entrant being present will be. In addition, the competition when n is larger is more fierce. Therefore, to maximize the expected total effort, the organizer should make the minimum prize for the scenario with larger number of entrants as small as possible. To facilitate this, for $n_1 > n_2$, as long as the minimum prize is positive in scenario n_1 and the minimum prize in scenario n_2 is smaller than $1/n_2$,[7] she should keep on lowering the minimum prize in scenario n_1 and increasing the minimum prize in scenario n_2. This is the principle of *cross-scenario prize transfer*: transferring prize from the scenario with higher number of entrants to the scenario with lower number of entrants elicits more effort in the presence of participation constraint.

Therefore, for each entry threshold, at the optimum, there is a unique number n^* such that the minimum prize is zero for $n > n^*$, the minimum prize is simply $1/n$ for $n < n^*$, and for $n = n^*$ the minimum prize can be any value in $[0, 1/n^*]$, which is determined by the binding participation constraint for the threshold type. In other words, at the optimum, the designer elicits the most effort or extracts (the most) surplus when $n > n^*$, and she completely gives up such opportunity (elicits zero effort) when $n < n^*$ to subsidize entry. It is clear that n^* decreases (weakly) in the entry threshold, as the organizer has to give up extracting effort more often if she wishes to induce

[7] The organizer's initial prize budget is normalized as 1.

lower threshold type to participate.

With the optimal prize allocation rule ready for any fixed entry threshold, the last step is to pin down the optimal entry threshold. The organizer faces the trade-off between extracting higher effort and inducing more entry. The organizer needs to find the optimal way of balancing these two effects. Quite surprisingly, it turns out that she always favors ex post effort extraction such that the optimal entry threshold is induced when the minimum prize is 0 for all $n \geq 2$ and it is 1 when $n = 1$.

We thus establish that the optimal prize allocation rule with costly entry is a winner-take-all regardless of the number of entrants. This result generalizes the winner-take-all principle in Moldovanu and Sela [9] into the environment with entry costs,[8] which are widely present in reality and thus need to be taken into account by the designer. However, our result implies that the organizer can design the allocation rule as if there were no entry cost at all when the cost function is linear or concave. This helps explain the prominence of winner-take-all in real world contests where the contestants usually have entry costs and the number of entrants is typically stochastic.

Our paper primarily belongs to the literature on optimal prize allocation in all-pay auction with incomplete information.[9] Our paper is most closely related to the seminal work of Moldovanu and Sela [9]. They establish that when the effort function is linear or concave, winner-take-all is optimal. Our paper confirms that the principle of winner-take-all is robust to costly entry. Minor [8] reexamines the same design problem as in Moldovanu and Sela [9] when contestants have convex costs of effort and when the contest designer has concave benefit of effort. Moldovanu and Sela [10] further investigate a two-stage all-pay auction framework, and Moldovanu, Sela, and Shi [11] study an environment where contestants care about their relative status. Moldovanu, Sela, and Shi [12] further accommodate carrots and sticks in their analysis.

A handful of papers study contests with a stochastic number of contestants. Higgins, Shughart, and Tollison [3], Kaplan and Sela [4], and Fu, Jiao, and Lu [2] investigate contests with costly entry. Higgins, Shughart, and Tollison [3] study the contest in which every rent seeker incurs a fixed cost to enter. They characterize the symmetric mixed-strategy zero-profit equilibrium in which the potential rent seekers participate randomly. Kaplan and Sela [4] also study contests where each potential contestant has a private cost of participating. They provide a rationale for the use of entry fee. Fu, Jiao, and Lu [2] consider the design of effort-maximizing contest with endogenous

[8] Moldovanu and Sela [9] show that, with free entry, winner-take-all is still optimal when the effort cost function is concave, and it may not be optimal when the effort cost function is convex. In section 4, we show that both results extend to the costly entry environment. When the cost function is convex, we also provide an example showing that winner-take-all is optimal when there is no cost of entry, while it is not optimal when there is a positive entry cost.

[9] There is another strand of literature which adopts the mechanism design approach to study the optimal contest design with incomplete information, which includes Kirkegaard [5], Polishchuk and Tonis [16], and Liu et al. [7], among others.

entry where a fixed pool of potential contestants bear a fixed entry cost to enter. The organizer can adjust the impact function of a generalized nested lottery contest as well as the prize allocation rule which can depend on the number of entrants. All these papers assume that the contestant's ability (cost of exerting effort) is common knowledge. Our paper differs from these studies by assuming the contestants' abilities are their own private information. There are some other papers studying exogenous stochastic entry. Myerson and Wärneryd [15] study contests when there are infinite many potential players and the set of players is a random variable. Münster [13], Lim and Matros [6], and Fu, Jiao, and Lu [1] consider a finite number of potential contestants. They assume that contestants participate with a fixed and independent probability, while in our paper the entry is endogenous.

The rest of this paper is organized as follows. In section 2, we set up the model. Section 3 analyzes the optimal contest rule. Discussions about concave cost function and convex cost function cases are in section 4. Section 5 concludes. Technical proofs are relegated to the appendix.

2 The Model

A risk neutral contest organizer has one dollar[10] to elicit effort from N (≥ 2) risk neutral potential contestants. For contestant i, his cost of exerting effort e_i is e_i/t_i, where t_i is his private information.[11,12] We assume that t_i's are independently and identically distributed with the cumulative distribution function $F(\cdot)$ on the support $[a, b]$ with $a > 0$. The probability density function $f(\cdot)$ is strictly positive on $[a, b]$.

Each contestant incurs a commonly known cost $c \in (0, 1/N]$ to participate in the contest.[13] Thus, a participant's payoff is equal to the prize he receives minus the cost of exerting effort and the entry cost c. The contestant's payoff is zero if he does not enter the contest. The organizer's goal is to design a contest rule to maximize the expected total effort using her budget.

Entry cost necessarily induces endogenous entry. For convenience, call the scenario that there are n entrants as *scenario* n.[14] A contest rule consists of two vectors $\mathbf{V}$ and $\mathbf{W}$, called the budget vector and prize allocation vector, respectively. Specifically, $\mathbf{V} = (V_1, V_2, \ldots, V_N) \in \mathbb{R}_+^N$ is the vector of sum of prizes so that $V_n \in [0, 1]$ is the sum of prizes in scenario n. $\mathbf{W} = (\mathbf{W}_1, \mathbf{W}_2, \ldots, \mathbf{W}_N)$ is the

[10] We can also assume that the organizer's budget is $V > 0$. Here "one dollar" is just a normalization.

[11] Moldovanu and Sela [9] assume that the cost is $e_i \cdot t_i$. Our model is equivalent to theirs.

[12] In section 4, we shall discuss the concave cost function and convex cost function cases.

[13] $c \leq 1/N$ is to ensure that the organizer can induce full entry if she wants to. This is a reasonable assumption as in reality the entry cost is often significantly smaller than the total prize. In fact, in Remark 3, we show that this assumption either is without loss of generality or rules out the uninteresting case that the contest cannot be held.

[14] In this paper, we use "participant" and "entrant" interchangeably. As we shall focus on symmetric equilibrium, the identities of the entrants are not important.

prize allocation vector, where $\mathbf{W}_n = (w_{n,1}, w_{n,2}, \ldots, w_{n,n}) \in \mathbb{R}^n_+$ with $w_{n,1} \geq w_{n,2} \geq \ldots \geq w_{n,n} \geq 0$ and $\sum_{j=1}^n w_{n,j} = V_n$ is the prize allocation vector in scenario n.[15] That is, $w_{n,j}$ is the prize for the j'th place winner in scenario n. Ties are broken randomly and fairly.

The timing of the game is as follows.

Time 0: $F(\cdot), c$ and N are revealed by nature as public information. Nature draws type for each contestant.

Time 1: The organizer chooses $\mathbf{V}$ and $\mathbf{W}$. And she commits to them. The contest rule is announced.

Time 2: All the potential contestants decide whether or not to participate in the contest. If a contestant decides to enter, he bears the entry cost and exerts effort without knowing the number of participants.[16]

Time 3: The prizes are allocated according to the rule announced at *time* 1.

Our model departs from that in Moldovanu and Sela [9] by assuming that the allocation of prizes can depend on the number of entrants and the budget in each scenario can vary. In the next section, we shall first begin with some analysis of the properties of the equilibrium.

3 Analysis of the Optimal Contest Rule

As the contest rule satisfies the anonymity property, it is natural to focus on symmetric equilibrium. We first study the behavior of the entrant's bidding function.

3.1 Equilibrium Analysis

Assume that (p, σ) is a symmetric (mixed strategy) Bayesian Nash equilibrium in the original game, where $p \in [0, 1]$ is the probability of entry.[17] That is, for contestant i with type t_i, $p(t_i)$ and $\sigma(\cdot|t_i)$ specify the probability of entry and a distribution function over $\mathbb{R}_+$ if he enters, respectively. Specifically, $\sigma(e|t)$ is the probability that the contestant with type t exerts effort less than or equal to e if he enters. Note that when $p = 0$, σ is not defined. Also note that if $p(t) > 0$ for some t, then $p(t') > 0$ for any $t' > t$ since the expected payoff is weakly increasing in type as required by

[15] Here we abuse the term a little to call both $\mathbf{W}$ and $\mathbf{W}_n$ as the "prize allocation vector".

[16] This means the contestant makes the entry decision and effort decision simultaneously. If the contestants observe the number of rival(s) after incurring the cost or the organizer announces the number of participants so that the participants exert effort after knowing the number of rival(s), the optimal contest rule remains the same. See Remark 1. Note that it is easy to see that observing the number of entrants is weakly dominated from the organizer's viewpoint as there are more constraints.

[17] The existence of equilibrium is not an issue since the bidding function derived later already gives an equilibrium.

incentive compatibility. Define $t_0 = \inf\{t : p(t) > 0\}$. Then for any $t > t_0$, $p(t) > 0$ and $\sigma(\cdot|t)$ is well defined.

Let $\tilde{V}(e)$ be the expected prize an entrant obtains when his opponents are using the strategy (p, σ). Denote the support of $\sigma(\cdot|t)$ as $S(t)$, then every $e \in S(t)$ yields the same expected payoff for the entrant with type t given the other contestants' strategies (p, σ). That is, for any $e_1, e_2 \in S(t)$, $\tilde{V}(e_1) - \frac{e_1}{t} = \tilde{V}(e_2) - \frac{e_2}{t}$. Note that the effort provision strategy σ is only valid for the entrant. The following four lemmas characterize the effort provision function.

It is intuitive that higher effort leads to higher prize. The following lemma confirms this. (All the proofs are in the appendix.)

Lemma 1 *Suppose $e_1, e_2 \in \cup_{t>t_0} S(t)$ and $e_1 < e_2$, then $\tilde{V}(e_1) \leq \tilde{V}(e_2)$.*

Our next result says that the support of different types cannot strictly overlap. In other words, their supports can only intersect at the boundary.

Lemma 2 *For any two types $t_1, t_2 > t_0$ with $t_1 < t_2$, there does not exist $e_1 \in S(t_1)$ and $e_2 \in S(t_2)$ such that $e_1 > e_2$.*

Lemma 3 *For any $t > t_0$, $S(t)$ is a singleton so that it is a function. Moreover, $S(t)$ is strictly increasing in $t > t_0$ or is equal to zero for all $t > t_0$.*

According to the above lemma, every entrant uses pure strategy, thus constitutes a bidding function. The bidding function is always zero when all the prizes are equal in any scenario that happens with positive probability (i.e., $w_{n,1} = w_{n,n}$ if scenario n happens with positive probability), while it is strictly increasing if there is some scenario (say scenario n) that happens with positive probability such that $w_{n,1} > w_{n,n}$. The reason for the latter case is simple: an infinitesimal increase in effort would yield a discrete jump in the expected prize obtained so that pooling would never happen. The following lemma further identifies the boundary condition of the bidding function.

Lemma 4 *Type t_0 must bid zero.*

We will focus on threshold entry in this paper. Threshold entry means that there exists a threshold t^c such that types higher than t^c enter with probability one, while types lower than it enter with zero probability. In fact, there is no loss of generality to focus on threshold entry. This results from Lemma 3 and the fact that higher type has higher expected payoff as required by incentive compatibility. Since the number of participants can be endogenous, the organizer can decompose the problem into two steps. She first fixes the entry threshold at t^c and solves for the

optimal contest rule. Then varying across all $t^c \in [a, b)$ will give her the optimal entry threshold and hence the optimal contest rule.[18] We shall first analyze the first step.

3.2 The Optimal Contest Rule for Fixed Threshold t^c

For threshold entry t^c, only contestants with type $t \geq t^c$ will enter. If $t^c > a$, then the entrant faces a stochastic number of rivals. An entrant knows that his rivals' types are independently drawn from the truncated CDF $G(t, t^c) = \frac{F(t)-F(t^c)}{1-F(t^c)}$, with density function $g(t, t^c) = \frac{f(t)}{1-F(t^c)}$, $t \in [t^c, b]$. According to Lemma 3, all entrants use pure strategies. We shall first derive the entrant's effort bidding function and the expected total effort inducible for any fixed $(\mathbf{V}, \mathbf{W})$,[19] and then we go on to pin down the optimal contest rule. To this end, we first derive the effort bidding function ignoring participation constraint for a fixed number of entrants. In other words, this is the bidding function when the entrant bids after knowing the number of rival(s). Although we assume that the entrants bid without observing the number of participant(s), as will be clear later (see Remark 1), since the contestants are risk neutral, the revelation policy of the number of entrants does not make a difference in the design of optimal prize allocation rule. Therefore, as will be shown later, the entrant's bidding function without the knowledge of the number of participant(s) is the weighted average of the bidding function with the knowledge of the number of participant(s) in each scenario, where the weight is the probability of the corresponding scenario happening from the perspective of the entrant.

3.2.1 The Bidding Function in the Hypothetical Situation

Suppose there are n entrants and the lowest type (i.e., t^c) bids zero, suppose further that the prize allocation vector is $\mathbf{W}_n = (w_{n,1}, w_{n,2}, \ldots, w_{n,n})$ with $w_{n,1} \geq w_{n,2} \geq \ldots \geq w_{n,n} \geq 0$ and $\sum_{j=1}^{n} w_{n,j} = V_n \in [0, 1]$, where $w_{n,j}$ is the prize for the j'th place winner. Consider the hypothetical situation (called *hypothetical situation* n) that the entrant bids *after* knowing that the number of entrants is n and that the entrant's participation constraint is ignored.[20] Although our interest is the bidding behavior when the entrant bids without knowing the number of rival(s), as we shall see later, the exercise here will help to understand the entrant's bidding behavior. We first derive the (unique) symmetric bidding function $e^{(n)}(t, \mathbf{W}_n, t^c)$ with $e^{(n)}(t^c, \mathbf{W}_n, t^c) = 0$ and the expected total effort in this hypothetical situation. The following lemma gives the characterization.

[18] Note that $t^c = b$ just means the organizer cancels the contest, which yields a total effort of zero.

[19] In fact, once the prize allocation vector $\mathbf{W}$ is given, the budget vector $\mathbf{V}$ is also uniquely pinned down. We keep the budget vector $\mathbf{V}$ as a choice variable to better describe the prize allocation rule and to facilitate our analysis of the optimum after introducing the notion of "minimum prize vector".

[20] This means the entrant cannot drop out of the contest once he is in.

Lemma 5 *In hypothetical situation n, given the prize allocation vector $\mathbf{W}_n = (w_{n,1}, w_{n,2}, \ldots, w_{n,n})$ with $w_{n,1} \geq w_{n,2} \geq \ldots \geq w_{n,n} \geq 0$,*

1) The unique symmetric bidding function $e^{(n)}(t, \mathbf{W}_n, t^c)$ satisfying $e^{(n)}(t^c, \mathbf{W}_n, t^c) = 0$ is given by

$$e^{(n)}(t, \mathbf{W}_n, t^c) = tV^{(n)}(t) - \int_{t^c}^{t} V^{(n)}(s)ds - t^c w_{n,n},$$

where

$$V^{(n)}(t) = \sum_{j=1}^{n} w_{n,n+1-j} \binom{n-1}{j-1} G^{j-1}(t, t^c)(1 - G(t, t^c))^{n-j}$$

is the expected prize an entrant with type t obtains.

2) The corresponding expected total effort is given by

$$TE^{(n)}(\mathbf{W}_n, t^c) = n \int_{t^c}^{b} J(t)V^{(n)}(t)g(t, t^c)dt - nt^c w_{n,n},$$

where $J(t) = t - \frac{1-F(t)}{f(t)}$.

Having equipped with the expressions for the unique symmetric bidding function and the expected total effort under any prize allocation vector $\mathbf{W}_n$ in the hypothetical situation, we are ready to derive the symmetric bidding function for a contestant with type $t \geq t^c$ and who is not aware of the number of entrants. Here we emphasize that the above lemma servers only as a tool to derive the bidding function of a contestant who decides to enter. It does not imply that the entrant knows which scenario he is in.

3.2.2 Entrants' Bidding Function

Suppose that a contestant's type is above the threshold so that he will enter. We need to derive his bidding strategy when he is uncertain of the number of rivals. According to Lemma 3, he will use pure strategy. To make a bid, the entrant can go through the following reasoning: "I first incur an entry cost c, which is already sunk. If I am in scenario n where the prize allocation vector is $\mathbf{W}_n$, and I am forced to stay in the contest, then if the other $n-1$ rivals are bidding according to $e^{(n)}(t, \mathbf{W}_n, t^c)$, then it is optimal for me to bid according to $e^{(n)}(t, \mathbf{W}_n, t^c)$ as well by Lemma 5. I also understand that the probability that I am in scenario n is $p_n(t^c) = \binom{N-1}{n-1} F^{N-n}(t^c)(1-F(t^c))^{n-1}$ so that bidding according to $e(t, \mathbf{W}, t^c) = \sum_{n=1}^{N} p_n(t^c) e^{(n)}(t, \mathbf{W}_n, t^c)$ would be my best choice if others are doing so". The following proposition shows that $e(t, \mathbf{W}, t^c)$ is indeed the (unique) symmetric bidding function.

Proposition 1 *Suppose the prize allocation vector is $\mathbf{W} = (\mathbf{W}_1, \mathbf{W}_2, \ldots, \mathbf{W}_N)$ and the entry threshold is t^c, where $\mathbf{W}_n = (w_{n,1}, w_{n,2}, \ldots, w_{n,n})$, $1 \leq n \leq N$, then*

1) The unique symmetric bidding function is $e(t, \mathbf{W}, t^c) = \sum_{n=1}^{N} p_n(t^c) e^{(n)}(t, \mathbf{W}_n, t^c)$*, where* $p_n(t^c) = \binom{N-1}{n-1} F^{N-n}(t^c)(1 - F(t^c))^{n-1}$ *and* $e^{(n)}(t, \mathbf{W}_n, t^c)$ *is given in Lemma 5.*

2) The expected total effort is $TE(\mathbf{W}, t^c) = \sum_{n=1}^{N} \binom{N}{n} (1 - F(t^c))^n F^{N-n}(t^c) TE^{(n)}(\mathbf{W}_n, t^c)$*, where* $TE^{(n)}(\mathbf{W}_n, t^c)$ *is given in Lemma 5.*

The entrant's bidding function can be equivalently seen as a weighted average of his individual bidding function in each scenario, where the weight is the probability of that scenario happening from his perspective. In other words, we may understand it as a two-step procedure. In the first step, the entrant first figures out his bidding function if he is in scenario n where his participation constraint is ignored, which is characterized in Lemma 5. In the next step, he decides on his effort level by taking a weighted average of all the N bidding functions.

Such kind of decomposition of the entrant's bidding function into scenario bidding ones will facilitate our analysis of the effects of the allocation of prizes in some scenario on the expected total effort. Note that such decomposition is not unique, however, the aggregate level $e(t, \mathbf{W}, t^c)$ must be unique. Note that $TE^{(n)}(\mathbf{W}_n, t^c)$, the *scenario n expected total effort*, is the expected total effort inducible *if* the entrants exert effort after knowing which scenario they are in. The expected total effort can be viewed as the weighted average of all these scenario expected total effort. However, we emphasize again that this only serves as a way of understanding the effects of prize allocation in every scenario on the expected total effort. It does not really mean that the total effort induced in scenario n is $TE^{(n)}(\mathbf{W}_n, t^c)$.

Remark 1 *Since the minimum prize in every scenario is non-negative, the entrant's expected payoff is non-negative after incurring the entry cost. Together with the fact that the contestants are risk neutral, we have the observation that the revelation policy of the number of participant(s) does not play a role in the design of optimal prize allocation rule. As can be seen from Proposition 1 above, the entrant's bidding function when he bids without knowing the number of rival(s) is a weighted average of the bidding function in each scenario when he bids after knowing the number of rival(s). As a result, from the organizer's perspective, revealing or concealing the number of participant(s) leads to the same expected total effort level ex ante.*

3.2.3 The Optimal Contest Rule for Fixed t^c

As mentioned above, we can treat the problem as if the entrants were making N individual decisions on effort levels for the N scenarios and then aggregating them. Now we are ready to analyze the organizer's problem

$$\max_{\mathbf{V},\mathbf{W}} TE(\mathbf{W}, t^c) = \sum_{n=1}^{N} \binom{N}{n} (1 - F(t^c))^n F^{N-n}(t^c) TE^{(n)}(\mathbf{W}_n, t^c)$$

subject to

$$\sum_{j=1}^{n} w_{n,j} = V_n \in [0, 1], \forall n; \tag{1}$$

$$u(t^c) = \sum_{n=1}^{N} p_n(t^c) w_{n,n} = c, \tag{2}$$

where $u(t^c)$ is the expected payoff of the threshold type t^c and $TE^{(n)}(\mathbf{W}_n, t^c)$ is given in Lemma 5.

Constraint (1) is the budget constraint. Constraint (2) is the participation constraint which says that the expected payoff for the threshold type must exactly offset the entry cost.[21] Note that since the bidding function is weakly increasing, incentive compatibility implies that all types higher than the threshold type must have an expected payoff no less than c.

For notation simplicity, we define $K_n = w_{n,n}$, which is called the *minimum prize* in scenario n. It is easy to see that $K_n = w_{n,n} \leq V_n/n$ since $w_{n,1} \geq w_{n,2} \geq \ldots \geq w_{n,n}$.

To solve for the optimal $(\mathbf{V}, \mathbf{W})$, we can first fix the the budget vector $\mathbf{V}$ and $w_{n,n}$ satisfying (2) to solve for the optimal prize allocation vector $\mathbf{W}$. And then we vary across all possible $\mathbf{V}$ and $w_{n,n}$ to obtain the optimum. In fact, when $w_{n,n}$ and the budget V_n for scenario n are both fixed for all n, maximizing $TE(\mathbf{W}, t^c)$ is equivalent to maximizing every $TE^{(n)}(\mathbf{W}_n, t^c)$ subject to (1). The following lemma characterizes the optimal scenario n prize vector.

Lemma 6 *The optimal solution to*

$$\max_{\mathbf{W}_n} TE^{(n)}(\mathbf{W}_n, t^c)$$

subject to

$$\sum_{j=1}^{N} w_{n,j} = V_n, \text{ with } w_{n,n} = K_n \leq \frac{V_n}{n}, \tag{3}$$

where K_n is a fixed number, is given by $w_{n,1} = V_n - (n-1)K_n$ and $w_{n,j} = K_n$ for all $j \geq 2$.

Lemma 6 echoes Moldovanu and Sela [9]. Denote the density of the i'th order statistics of the n i.i.d. random variables following CDF $G(t, t^c)$ as $g_{(i,n)}(t, t^c)$. It is well know that $g_{(i,n)}(t, t^c) = n\binom{n-1}{i-1} G^{i-1}(t, t^c)(1 - G(t, t^c))^{n-i} g(t, t^c)$. By Lemma 5, the expected total effort in scenario n can

[21] If $t^c > a$, then the threshold type t^c must be indifferent between entering and leaving so that his expected payoff is c. When $t^c = a$, $u(a) \geq c$ if and only if $w_{N,N} \geq c$. Then it reduces to full entry with minimum prize c. It is obvious that in this case $w_{N,N} = c$. Therefore, there is no loss to assume $u(t^c) = c$ for all t^c.

be expressed as a linear combination of the n prizes $w_{n,1}, w_{n,2}, \ldots, w_{n,n}$, minus a fixed number $nt^c w_{n,n} = nt^c K_n$,

$$TE^{(n)}(\mathbf{W}_n, t^c) = \sum_{j=1}^{n} w_{n,n+1-j} \int_{t^c}^{b} J(t) g_{(j,n)}(t, t^c) dt - nt^c w_{n,n}.$$

In the proof of Lemma 6, we show that the coefficient for $w_{n,j}$, $\int_{t^c}^{b} J(t) g_{(n+1-j,n)}(t, t^c) dt$, is decreasing in j and $\int_{t^c}^{b} J(t) g_{(n,n)}(t, t^c) dt > 0$. As a result, the first prize, $w_{n,1}$, is most effective in inducing the scenario n effort. Therefore, the organizer should make it as large as possible. This is the **cross-rank transfer**: transferring prizes from low ranks to enlarge the prize to be awarded to the highest rank so as to induce more effort. Constrained by (3), the lowest prize an entrant can have is K_n so that the prize allocation vector in the above lemma is optimal. Note that setting $K_n = 0$ goes back to the winner-take-all result in Moldovanu and Sela [9].

Equipped with Lemma 6, the highest expected total effort inducible under $\mathbf{V} = (V_1, V_2, \ldots, V_N)$ and $\mathbf{K} = (K_1, K_2, \ldots, K_N)$ can be written as[22]

$$TE(\mathbf{V}, \mathbf{K}, t^c) = N(1 - F(t^c)) \sum_{n=1}^{N} p_n(t^c) \int_{t^c}^{b} J(t)(V_n - nK_n) G^{n-1}(t, t^c) g(t, t^c) dt.$$

From here, it is easy to see that the expected total effort is strictly increasing in V_n so that at the optimum, $V_n = 1$ for all $n \geq 2$.[23] That is, the organizer will always exhaust her budget when there are at least two entrants.

Thus, the final step to solve the optimal contest rule for fixed threshold t^c is to find the optimal minimum prize vector $\mathbf{K}$. Formally, the organizer's problem now reduces to

$$\max_{\mathbf{K}} TE(\mathbf{K}, t^c) = N(1 - F(t^c)) \sum_{n=1}^{N} p_n(t^c) \int_{t^c}^{b} J(t)(V_n - nK_n) G^{n-1}(t, t^c) g(t, t^c) dt$$

subject to

$$V_n \leq 1, \forall n, \text{ with equality when } n \geq 2; \tag{4}$$

$$K_n \leq \frac{V_n}{n}, \forall n, \text{ with equality when } n = 1; \tag{5}$$

$$\sum_{n=1}^{N} p_n(t^c) K_n = c, \tag{6}$$

[22] For the derivation, please refer to the appendix.

[23] If $t^c = a$ so that only K_N matters, then the expected total effort is constant in V_n, $\forall n \leq N - 1$. However, in this case, there is no loss to assume that $V_n = 1$ for all $n \geq 2$ as it does not affect the constraints (5) and (6) below. When $n = 1$, $K_1 = V_1$ so that increasing V_1 does not affect the expected total efforts. Also note that changing V_1 will affect both (5) and (6) so that we cannot assume $V_1 = 1$.

where $p_n(t^c) = \binom{N-1}{n-1}(1-F(t^c))^{n-1}F^{N-n}(t^c)$ as defined before, is the probability that an entrant is in scenario n.

In order to describe the optimal solution, decompose the type interval $[a,b]$ into N disjoint intervals: $[a,b] = \cup_{n=1}^{N}[t_n, t_{n-1}]$ with $t_0 = b$ and $t_N = a$. For $n = 1, 2, \ldots, N-1$, t_n is the unique solution to the following equation[24]

$$\sum_{j=1}^{n} \frac{p_j(t^c)}{j} = c. \tag{7}$$

It is obvious that $a = t_N < t_{N-1} < t_{N-2} < \ldots < t_1 < t_0 = b$ since $\sum_{j=1}^{n} \frac{p_j(t^c)}{j}$ is strictly increasing in $t^c \in (a,b)$.[25] The following proposition characterizes the optimal minimum prize vector.

Proposition 2 *If $t^c \in [t_n, t_{n-1}]$ for some n, then the optimal minimum prize vector $\mathbf{K}^*(t^c) = (K_1^*(t^c), K_2^*(t^c), \ldots, K_N^*(t^c))$ is given by*[26,27]

$$K_m^*(t^c) = \begin{cases} \frac{1}{m}, \text{ if } m < n; \\ 0, \text{ if } m > n; \\ \frac{c - \sum_{i=1}^{n-1} \frac{p_i(t^c)}{i}}{p_n(t^c)}, \text{ if } m = n. \end{cases}.$$

And the optimal budget vector is $\mathbf{V}^*(t^c) = (V_1^*(t^c), V_2^*(t^c), V_3^*(t^c), \ldots, V_N^*(t^c)) = (K_1^*(t^c), 1, 1, \ldots, 1)$.

According the above proposition, when $t^c \in [t_n, t_{n-1}]$, $K_m^*(t^c) = 1/m$ for all $m < n$ and $K_m^*(t^c) = 0$ for all $m > n$. This means that when the number of entrants is smaller than n, all the entrants share the prize equally; when the number of entrants is larger than n, then it is a winner-take-all in light of Lemma 6. The intuition of Proposition 2 is as follows. We mentioned earlier that although each entrant makes his effort level decision without knowing the number of rivals, he could plan his effort *as if* he had N bidding functions, with one for each scenario. Hypothetically, this would induce N scenario total efforts. As such, the expected total effort from the organizer's viewpoint can be regarded as a weighted average of these N scenario total efforts.[28] Lowering K_n (i.e., lowering the lowest prize $w_{n,n}$ in scenario n) will induce more scenario n effort

[24] Note that the left hand side of the equation is continuous in $t^c \in [a,b]$. When $n \leq N-1$, $\sum_{j=1}^{n} \frac{p_j(a)}{j} = 0 < c$; $\sum_{j=1}^{n} \frac{p_j(b)}{j} = p_1(b) = 1 > c$. Therefore, there must exist one solution. For the uniqueness, note that $\frac{d}{dt^c}(\sum_{j=1}^{n} \frac{p_j(t^c)}{j}) = \frac{f(t^c)}{1-F(t^c)}(p_{n+1}(t^c) + \sum_{j=2}^{n} \frac{p_j(t^c)}{j}) > 0$, when $t^c \in (a,b)$. (When $n = 1$, $\sum_{j=2}^{n} \frac{p_j(t^c)}{j}$ is defined as zero.) For the derivation details, see the proof of Lemma 7.

[25] See footnote 24.

[26] When $t^c = a$, $p_n(t^c) = 0$ for all $n \leq N-1$ so that only K_N matters. In this case, the optimal K_n for $n \leq N-1$ may not be unique as they do not enter the objective function. However, when $t^c > a$, the optimal minimum prize vector $\mathbf{K}$ is unique.

[27] $\sum_{i=k_1}^{k_2} \phi(i)$ is defined as zero if $k_1 > k_2$, where $\phi(\cdot)$ is any function.

[28] This can also be understood from the irrelevance of the revelation of the number of entrants. We mentioned before that revealing the number of entrants yields the same expected total effort as that induced by concealing the

by the **cross-rank transfer** as the first prize in scenario n ($w_{n,1}$) can now be further enhanced. However, the organizer also needs to take care of the threshold type's participation constraint. A decrease of K_n must be compensated by an increase of K_m for some $m \neq n$. However, an increase in K_m will result in less scenario m total effort again due to the cross-rank transfer. Note that a decrease of K_n by $\varepsilon/p_n(t^c)$ must be compensated with an increase of K_m by $\varepsilon/p_m(t^c)$ to keep the participation constraint balanced.[29] Therefore, she faces a trade-off: an increase of total effort in some scenario comes along with a corresponding drop of total effort in another scenario. Then which scenario should she sacrifice? It turns out that the higher the number of entrants n is, the more effective it is to induce scenario total effort using one penny weighted by the reverse of the probability of that scenario happening from an entrant's perspective (i.e., $\varepsilon/p_n(t^c)$). This is rather intuitive as when the number of entrants is larger, the expectation of the highest order statistics is higher, i.e., the probability of a high type entrant being present is higher, and the competition among entrants are more fierce. Therefore, the organizer should make K_n small for large enough n. This is the **cross-scenario transfer**: transferring prize from scenario m_1 to scenario $m_2 < m_1$ in order to induce more expected total effort. The organizer should keep making such transfers as long as there are m and m' with $m > m' > 1$ such that $K_m > 0$ and $K_{m'} < 1/m'$. This is because in this case she can transfer prize from scenario m to scenario m' to further elicit effort.

Thus, in order to subsidize the participants to induce entry for the threshold t^c, the most effective way for the organizer is to first fulfil the scenarios with smaller number of entrants. Specifically, she first set $K_1 = 1$ to check whether the participation constraint (6) is satisfied. If it already exceeds the cost, then she does not need to exhausts her budget; if it cannot cover the cost c, then she goes on to set $K_2 = 1/2$, i.e., let the two entrants share the prize. If this exceeds the cost, then she can set a lower K_2, i.e., the first prize is larger than the second one when there are two entrants; if it does not cover the cost, then she continues to set $K_3 = 1/3$ to repeat the above procedure until the cost is covered.

One interesting cutoff point deserves some attention. When $t^c = t_1$, the budget vector is $(1, 1, \ldots, 1)$ and the minimum prize vector is $(0, 0, \ldots, 0)$. This means that in every scenario it is a winner-take-all with total prize 1. The contest rule is independent of the number of entrants. In the next subsection, we shall show that this is indeed the optimum.

number of entrants *ex ante*. Therefore, the expected total effort when the number of entrants is concealed can be seen as the weighted average of the expected total effort in each scenario when the number of entrants is revealed before the entrants make their bids.

[29] Note that $p_j(t^c) \neq 0$ when $t^c \in (a, b)$ for any j with $1 \leq j \leq N$. When $t^c = a$, i.e., full entry, only K_N is relevant.

3.3 The Optimal Contest Rule

Having characterized the optimal contest rule for any fixed entry threshold, we are now ready to further identify the optimal entry threshold, hence the optimal contest rule. By Lemma 6 and Proposition 2, for each fixed t^c, we have the highest expected total effort inducible which is denoted as $TE^*(t^c)$.

When choosing the optimal threshold t^c, the organizer faces the following trade-off between extracting surplus and encouraging entry. On the one hand, when t^c is small, the expected number of entrants is large and the probability of a large number of entrants is high. However, from Proposition 2, we see that she also has to give up extracting rents when the number of entrants is smaller than a certain cutoff. One the other hand, when t^c is high, the expected number of entrants is small and the probability of a large number of participants is low. However, now she is able to extract (the most) surplus even when the number of entrants is small, which happens with relatively higher probability. With these two conflicting effects, it is not obvious which one dominates. The following Lemma shows that the organizer always favors surplus extraction.

Lemma 7 *$TE^*(t^c)$ is strictly increasing in $[a, t_1]$ and strictly decreasing in $[t_1, b)$.*

It is quite intuitive that the expected total effort is strictly decreasing in $[t_1, b)$. Increasing the threshold above t_1 means decreasing the prize for the single entrant, which will discourage entry given that the minimum prizes in all other scenarios are always zero. However, when the induced threshold $t^c \geq t_1$, all the minimum prizes for scenarios $n \geq 2$ are zero so that the organizer extracts the most surplus *ex post*. Thus, the trade-off between surplus extraction and inducing entry disappears: increasing the threshold above t_1 will only discourage entry without extracting more surplus *ex post*. As a result, it is never optimal to raise the induced entry threshold above t_1. When the entry threshold is lower than t_1, the trade-off does exist and Lemma 7 implies that the effect of surplus extraction is always the dominant one.

Lemma 7 implies that the optimal cutoff is t_1. As mentioned before, when $t^c = t_1$, the organizer exhausts her budget in every scenario and the minimum prize is zero in each scenario. This is the following proposition.

Proposition 3 *The optimal contest rule with entry cost is winner-take-all with prize 1, which is independent of the number of entrants.*

Remark 2 *As a direct consequence of Remark 1, winner-take-all is still optimal when the participants exert effort after knowing the number of rival(s).*

In reality, entry cost is widely present and cannot be ignored. The optimal design of a contest prize allocation rule should take this into account. However, our result implies that the organizer can design the allocation rule as if there were no entry cost at all. The winner-take-all result when there is no entry cost in Moldovanu and Sela [9] is still optimal. Note that the only case that the organizer cannot extract surplus is when there is only one entrant, which happens with probability $F^{N-1}(t_1) = c$. Recall that we have normalized the total budget as one. In reality, the magnitude of entry cost is significantly smaller than the total prize so that the probability of single entrant is extremely small. This helps explain the prominence of winner-take-all in the real world where the contestants usually have entry cost and the number of entrants is typically stochastic.

Remark 3 *Recall that in the model setup we assume that $c \leq 1/N$. In footnote 13 we mention that this assumption either is without loss of generality or rules out the uninteresting case that the contest cannot be held. This is the consequence of Proposition 2 and Lemma 7. The rough idea is that when $c \in (1/N, 1)$, there is a certain cutoff type such that all entry thresholds lower than it cannot be supported. However, any type higher than it is inducible by the prize allocation rule in Proposition 2, which yields an expected payoff less than that by winner-take-all according to Lemma 7. When $c \geq 1$ there will be no entrant as the entry cost exceeds the organizer's budget so that she cannot subsidize the entrants. Detailed arguments are provided in the appendix.*

4 Discussions

We have focused on linear cost function in the main analysis. Moldovanu and Sela [9] show that, with free entry, winner-take-all is still optimal when the cost function is concave, while it may not be optimal when the cost function is convex. Both results can be extended to the costly entry environment.

Suppose that the contestant's cost of exerting effort e when his type is t is $\tilde{h}(e)/t$, where $\tilde{h}(0) = 0$, $\tilde{h}' > 0$, and $\tilde{h}'' \leq 0$. When the effort cost function is concave, winner-take-all with prize 1 is still optimal. The intuition is as follows. Recall that the three crucial driving forces of the optimality of winner-take-all: the cross-rank transfer, the cross-scenario transfer, and the strict dominance of surplus extraction over encouraging entry. Under concave cost function assumption, the marginal cost of exerting effort is diminishing so that it is even more efficient to motivate the strong contestant to exert more effort. Therefore, the cross-rank transfer still applies as now the organizer can extract even more surplus by transferring prizes from low ranks to the highest rank. Similarly, as the marginal cost is diminishing, the cross-scenario transfer still holds. Finally, the organizer favors surplus extraction even more than she does in the linear cost case. As a result, the surplus extraction effect still strictly dominates the encouraging entry effect. In the appendix we

provide a detailed proof of the optimality of winner-take-all in the concave cost function case.

When the effort cost function is convex, several positive prizes may be optimal in the free entry environment, as shown in Moldovanu and Sela [9]. It is easy to see that when entry is costly, winner-take-all may not be optimal. However, unlike the linear cost case and the concave cost case where winner-take-all is robust, with convex cost it is possible that winner-take-all is optimal when there is no cost of entry, while it is not optimal when there is a positive entry cost. We provide such an example in the appendix. This illustrates how the introduction of entry cost will affect the optimal allocation of prizes.

5 Concluding Remarks

In this paper, we accommodate costly entry of players and study the effort-maximizing prize allocation rule in all-pay auctions with incomplete information. We identify two effects of prize transfers that are crucial for characterizing the optimal prize structure: the cross-rank prize transfer within scenario and the cross-scenario prize transfer. The cross-rank transfer within scenario echoes the insight in Moldovanu and Sela [9] that the first prize is most effective in inducing effort. The cross-scenario transfer reveals how the organizer should optimally balance the trade-off between eliciting effort and inducing entry by adjusting jointly all prizes across scenarios.

The main finding is that winner-take-all remains optimal in such environments when effort cost function is linear or concave, which confirms the robustness of "winner-take-all" principle first established in Moldovanu and Sela [9].

Although the prize allocation rule can depend on the number of entrants, at the optimum, the organizer adopts a very simple contest rule. In other words, the organizer can design the prize allocation rule as if there were no entry costs. This justifies the prevalence of winner-take-all in contests in spite of the ubiquity of entry costs in reality.

6 Appendix

6.1 Proofs of the Linear Cost Case

Proof of Lemma 1: If there exists some $t > t_0$ such that $e_1, e_2 \in S(t)$, then since $e_1 < e_2$, and since e_1 and e_2 lead to the same expected payoff for the entrant with type t, $\tilde{V}(e_1) < \tilde{V}(e_2)$. Now assume that $e_1 \in S(t_1)$ and $e_2 \in S(t_2)$ with $e_1, e_2 \notin S(t_1) \cap S(t_2)$, where $t_1, t_2 > t_0$. Incentive compatibility requires that for type t_2 contestant exerting effort e_2 is weakly better than exerting effort e_1. That is, $\tilde{V}(e_2) - \frac{e_2}{t_2} \geq \tilde{V}(e_1) - \frac{e_1}{t_2}$. If $\tilde{V}(e_1) > \tilde{V}(e_2)$, then exerting effort e_1 is strictly

better. A contradiction. $\square$

Proof of Lemma 2: Suppose to the contrary that there exists $e_1 \in S(t_1)$ and $e_2 \in S(t_2)$ such that $e_1 > e_2$. By lemma 1, $\tilde{V}(e_1) \geq \tilde{V}(e_2)$. Incentive compatibility requires that for type t_1 exerting effort e_1 is weakly better than exerting effort e_2. That is, $\tilde{V}(e_1) - \frac{e_1}{t_1} \geq \tilde{V}(e_2) - \frac{e_2}{t_1}$, which implies that $\tilde{V}(e_1) - \tilde{V}(e_2) \geq \frac{e_1 - e_2}{t_1}$. Similarly, incentive compatibility requires that for type t_2 exerting effort e_2 is weakly better than exerting effort e_1. That is, $\tilde{V}(e_2) - \frac{e_2}{t_2} \geq \tilde{V}(e_1) - \frac{e_1}{t_2}$, which implies that $\tilde{V}(e_1) - \tilde{V}(e_2) \leq \frac{e_1 - e_2}{t_2}$. However, since $\tilde{V}(e_1) - \tilde{V}(e_2) \geq \frac{e_1 - e_2}{t_1}$ and since $t_1 < t_2$, $\tilde{V}(e_1) - \tilde{V}(e_2) \leq \frac{e_1 - e_2}{t_2}$ is impossible. $\square$

Proof of Lemma 3: First notice that if for any scenario which happens with positive probability, all the prizes in that scenario are equal, then obviously $S(t)$ only contains zero since there is no incentive for the entrant to exert a positive effort as the prize associated with any rank is always the same within a scenario. In this case, $S(t)$ is trivially a singleton which means every entrant uses a pure strategy. Now if there is some scenario, say scenario n, which happens with positive probability, such that $w_{n,i} > w_{n,j}$ for some i, j with $1 \leq i < j \leq n$. Then in this case, pooling can never happen. Pooling means that there is some effort level e such that $e \in S(t)$ for $t \in T'$ where T' has a positive measure. This is because if that is the case, then ties happen with positive probability. However, since $w_{n,i} > w_{n,j}$, an entrant with type $t \in T'$ can increase effort e by an infinitesimal amount such that the effort cost is negligible but the expected prize has a discrete jump. Since pooling can never happen, it follows that ties happen with zero probability. This in turn implies that every entrant uses a pure strategy. This is because if there exists some $t > t_0$ such that $S(t)$ contains at least two points, say e_1 and e_2. Without loss of generality, assume that $e_1 < e_2$. However, since ties happen with zero probability, by Lemma 2 exerting effort e_1 yields the same expected prize as e_2 (i.e., $\tilde{V}(e_1) = \tilde{V}(e_2)$). This implies that exerting effort e_1 is strictly better. A contradiction. Thus, $S(t)$ is a singleton so that it is a function. This also means that every entrant uses a pure strategy. Note that $S(t)$ must be strictly increasing. To see this, first note that it must be weakly increasing by Lemma 2. Suppose there exist t_1, t_2 with $t_1 < t_2$ such that $S(t_1) = S(t_2)$. Then for any $t \in [t_1, t_2]$, $S(t) = S(t_2)$. This means that there is pooling when $t \in [t_1, t_2]$. A contradiction. $\square$

Proof of Lemma 4: We only need to consider the case that the bidding function is strictly increasing. In this case, type t_0 is the infimum of types that will enter the contest so that such type entrant must always obtain the lowest prizes in every scenario in any symmetric strictly increasing bidding function. Therefore, if the t_0 type exerts a strictly positive effort, then by exerting zero effort his expected prize does not change, while he saves the effort cost. This would violate the incentive compatibility. Therefore, the t_0 type must exert zero effort. $\square$

Proof of Lemma 5: First note that if $w_{n,1} = w_{n,2} = \ldots = w_{n,n}$, then $e^{(n)}(t, \mathbf{W}_n, t^c) = 0$ for all $t \geq t^c$ so that 1) and 2) in Lemma 5 are trivially satisfied.

Now assume $w_{n,1} > w_{n,n}$. Then similar to the proof of Lemma 3, the symmetric bidding function is strictly increasing. To derive the bidding function and total effort, we adopt an approach similar to the revenue equivalence result in auctions.

Suppose each of the n entrants reports his type to the contest designer. This constitutes a report profile $\mathbf{t}$. The mechanism consists of $\mathbf{v}(\mathbf{t})$ and $\mathbf{e}(\mathbf{t})$, where $\mathbf{v}(\mathbf{t}) = (v_1(\mathbf{t}), v_2(\mathbf{t}), \ldots, v_n(\mathbf{t})) \in \mathbb{R}^n$ and $\mathbf{e}(\mathbf{t}) = (e_1(\mathbf{t}), e_2(\mathbf{t}), \ldots, e_n(\mathbf{t})) \in \mathbb{R}^n_+$ so that $v_i(t_i, \mathbf{t}_{-i})$ and $e_i(t_i, \mathbf{t}_{-i})$ is the prize and effort for entrant i when he reports t_i.

Define the expected prize of entrant i with report $\tilde{t}_i$ as

$$\tilde{V}_i^{(n)}(\tilde{t}_i) = \int_{\mathbf{t}_{-i}} v_i(\tilde{t}_i, \mathbf{t}_{-i}) \mathbf{g}_{-i}(\mathbf{t}_{-i}, t^c) d\mathbf{t}_{-i},$$

where $\mathbf{t}_{-i} = (t_1, ..., t_{i-1}, t_{i+1}, ..., t_n)$ and $\mathbf{g}_{-i}(\mathbf{t}_{-i}, t^c)$ denotes the density of $\mathbf{t}_{-i} \in [t^c, b]^{n-1}$.

Given that other entrants truthfully report their types, entrant i's expected payoff when reporting $\tilde{t}_i$ is

$$u_i(\tilde{t}_i, t_i) = \tilde{V}_i^{(n)}(\tilde{t}_i) - \frac{\int_{\mathbf{t}_{-i}} e_i(\tilde{t}_i, \mathbf{t}_{-i}) \mathbf{g}_{-i}(\mathbf{t}_{-i}, t^c) d\mathbf{t}_{-i}}{t_i}.$$

Incentive compatibility constraint reads

$$u_i(t_i, t_i) \geq u_i(\tilde{t}_i, t_i), \forall \tilde{t}_i, t_i, \forall i. \tag{8}$$

Define $\tilde{u}_i(\tilde{t}_i, t_i) = t_i \cdot u_i(\tilde{t}_i, t_i)$. Then

$$\tilde{u}_i(\tilde{t}_i, t_i) = t_i \tilde{V}_i^{(n)}(\tilde{t}_i) - \int_{\mathbf{t}_{-i}} e_i(\tilde{t}_i, \mathbf{t}_{-i}) \mathbf{g}_{-i}(\mathbf{t}_{-i}, t^c) d\mathbf{t}_{-i}.$$

From (8) and the Envelope Theorem, we have

$$\frac{d\tilde{u}_i(t_i, t_i)}{dt_i} = \frac{\partial \tilde{u}_i(\tilde{t}_i, t_i)}{\partial t_i}|_{\tilde{t}_i = t_i} = \tilde{V}_i^{(n)}(t_i),$$

which leads to

$$\tilde{u}_i(t_i, t_i) - \tilde{u}_i(t^c, t^c) = \int_{t^c}^{t_i} \tilde{V}_i^{(n)}(s) ds.$$

Standard derivations such as those in Myerson [14] lead to that IC holds if and only if the following two conditions are satisfied:

$$\int_{\mathbf{t}_{-i}} e_i(t_i, \mathbf{t}_{-i}) \mathbf{g}_{-i}(\mathbf{t}_{-i}, t^c) d\mathbf{t}_{-i} = t_i \tilde{V}_i^{(n)}(t_i) - \int_{t^c}^{t_i} \tilde{V}_i^{(n)}(s) ds - t^c \cdot u_i(t^c, t^c), \forall t_i, \forall i, \tag{9}$$

$$\tilde{V}_i^{(n)}(t_i') \geq \tilde{V}_i^{(n)}(t_i), \ \forall t_i' > t_i, \forall i. \tag{10}$$

As a result, the expected total effort is[30]

$$\int_{\mathbf{t}\in[t^c,b]^n} \sum_i e_i(\mathbf{t})\mathbf{g}(\mathbf{t},t^c)d\mathbf{t} = \sum_i \int_{t^c}^b J(t_i)\tilde{V}_i^{(n)}(t_i)g(t_i,t^c)dt_i - \sum_i t^c u_i(a,a).$$

Now we are ready to prove the lemma. Note that the contest game is a special mechanism so that it must satisfy the IC constraint. As a result, the bidding function which is independent of rivals' reports in the contest game can be expressed as

$$e_i^{(n)}(t,\mathbf{W}_n,t^c) = t_i\tilde{V}_i^{(n)}(t_i) - \int_{t^c}^{t_i} \tilde{V}_i^{(n)}(s)ds - t^c \cdot u_i(t^c,t^c).$$

Since the bidding function is strictly increasing, ties happen with zero probability so that type t^c's expected prize $\tilde{V}_i^{(n)}(t^c) = w_{n,n}$. Combined with the assumption that type t^c exerts zero effort, we have $u_i(t^c,t^c) = w_{n,n}$. Thus, it suffices to characterize the expected prize $\tilde{V}_i^{(n)}(t_i)$ for all i to derive the bidding function and the expected total effort.

Recall that when $w_{n,1} > w_{n,n}$, any symmetric equilibrium is pure and strictly increasing. As a result, for entrant i who uses type t_i's strategy, his probability of being the $n+1-j$'th place winner (i.e., there are $n-j$ entrants' types are higher than his and $j-1$ entrants' types are lower than his) is $\binom{n-1}{j-1}G^{j-1}(t_i,t^c)(1-G(t_i,t^c))^{n-j}$, where $1 \le j \le n$. Thus, the expected prize for reporting t_i is

$$\tilde{V}_i^{(n)}(t_i) = \sum_{j=1}^n w_{n,n+1-j}\binom{n-1}{j-1}G^{j-1}(t_i,t^c)(1-G(t_i,t^c))^{n-j}.$$

Note that $\tilde{V}_i^{(n)}(t) = \tilde{V}_j^{(n)}(t)$ by symmetry so that we can drop the subscript. That is, the expected prize for a type t entrant is given by

$$\tilde{V}^{(n)}(t) = \sum_{j=1}^n w_{n,n+1-j}\binom{n-1}{j-1}G^{j-1}(t,t^c)(1-G(t,t^c))^{n-j} = V^{(n)}(t).$$

Therefore, the bidding function is

$$e^{(n)}(t,\mathbf{W}_n,t^c) = tV^{(n)}(t) - \int_{t^c}^t V^{(n)}(s)ds - t^c w_{n,n},$$

and the total effort can be expressed as

$$n\int_{t^c}^b J(t)V^{(n)}(t)g(t,t^c)dt - nt^c w_{n,n}.$$

Moreover, the uniqueness of the bidding function is obvious since it is uniquely pinned down by the expected prize function, which is unique. □

[30] Note that $J(t) = t - \frac{1-F(t)}{f(t)} = t - \frac{1-G(t,t^c)}{g(t,t^c)}$, $t \ge t^c$.

Proof of Proposition 1: Note that if for any n such that $p_n(t^c) > 0$, $w_{n,1} = w_{n,n}$, then $e^{(n)}(t, \mathbf{W}_n, t^c) = 0$ as in the proof of Lemma 5. In this case, Proposition 1 trivially holds. Now assume that for some n such that $p_n(t^c) > 0$, $w_{n,1} > w_{n,n}$, then by the proof of Lemma 3, the bidding function is strictly increasing. The proof of this case is similar to that of Lemma 5. By Lemma 3, every entrant uses pure strategy. Assume that $e(t, \mathbf{W}, t^c)$ is a symmetric bidding function under the prize allocation vector $\mathbf{W}$. By Lemma 4, the threshold type t^c entrant must bid zero, i.e., $e(t^c, \mathbf{W}, t^c) = 0$. To derive the symmetric bidding function and show its uniqueness, we shall also use the same idea as revenue equivalence in auctions.

We first look at a general contest mechanism as in the proof of Lemma 5. Since the number of entrants is random, we make use of the semirevelation principle (Stegeman [18]). According to that, there is no loss to focus on semidirect mechanisms where only the participants are required to reveal his type to the designer, while those who do not participate are not required to send messages. And then given the reports, the mechanism specifies the prize and effort for each contestant. Our goal here is to use the incentive compatibility constraint in the semidirect mechanism to express the effort function in terms of the prize allocation rule.

Following Stegeman [18], we use a null message $\emptyset$ to denote the signal of nonparticipation. There is no loss to confine the message space to $[a, b] \cup \{\emptyset\}$ for every contestant. Denote $\tilde{m}_i(t_i, 1)$ as the message contestant i with type t_i reports when he participates, where "1" means participation. Similarly, $\tilde{m}_i(t_i, 0)$ is the message contestant i with type t_i sends when he does not participate, where "0" means nonparticipation. For convenience, $\tilde{m}_i(t_i, 0) = \emptyset$. A contestant's strategy $\pi_i(t_i)$ can be expressed as $(q_i(t_i), \tilde{m}_i(t_i, 1))$, where $q_i(t_i) \in [0, 1]$ is the probability of entry. Denote $\pi = (\pi_1, \pi_2, \ldots, \pi_N)$ as the strategy profile. π is *semidirect* if $\tilde{m}_i(t_i, 1) = t_i$, $\forall t_i \in [a, b], \forall i \in \mathcal{N}$, where $\mathcal{N}$ is the set of potential contestants. Let $\tilde{\mathbf{s}} = (\tilde{s}_1, \ldots, \tilde{s}_N)$ denote the message profile reported. A mechanism can be denoted by $(\mathbf{v}(\tilde{\mathbf{s}}), \mathbf{e}(\tilde{\mathbf{s}}))$, where $\mathbf{v}(\tilde{\mathbf{s}}) = (v_1(\tilde{\mathbf{s}}), \cdots, v_N(\tilde{\mathbf{s}}))$ and $\mathbf{e}(\tilde{\mathbf{s}}) = (e_1(\tilde{\mathbf{s}}), \cdots, e_N(\tilde{\mathbf{s}}))$ are the prize allocation function and effort function, respectively. So $v_i(\tilde{\mathbf{s}})$ and $e_i(\tilde{\mathbf{s}})$ are the prize and effort for contestant i, respectively.[31]

As in Stegeman [18], we call $\theta = (\pi, \mathbf{v}, \mathbf{e})$ a procedure. It is semidirect if π is semidirect. According to the semirevelation principle in Stegeman [18], there is no loss of generality to restrict attention to semidirect procedures (mechanisms) in which it is an equilibrium for every participant to truthfully reveal his type, while the nonparticipant is not required to submit a signal. Since we focus on threshold entry at t^c, we can further restrict the message space to $[t^c, b] \cup \{\emptyset\}$ for every contestant. Threshold-entry means $\pi_i(t_i) = (1, t_i)$ if $t_i \in [t^c, b]; \pi_i(t_i) = (0, \emptyset)$, if $t_i \in [a, t^c)$. For notation simplicity, let $m_i(t_i)$ be the message sent by contestant i with type t_i in the semidirect

[31] Note that the specification here allows the effort required by the mechanism varies with the number of entrants which covers the case that the entrant's effort provision is uniform across all scenarios. Our interest here is to use the IC in the semidirect mechanism to rewrite the effort function in terms of the prize allocation rule.

mechanisms. Therefore, $m_i(t_i) = t_i$, if $t_i \in [t^c, b]; m_i(t_i) = \emptyset$, if $t_i \in [a, t^c)$.

Define the expected prize of contestant i with type $\tilde{t}_i$:

$$\tilde{V}_i(\tilde{t}_i) = \int_{\mathbf{t}_{-i}} v_i(m_i(\tilde{t}_i), \mathbf{m}_{-i}(\mathbf{t}_{-i}))\mathbf{f}_{-i}(\mathbf{t}_{-i})d\mathbf{t}_{-i},$$

where $\mathbf{t}_{-i} = (t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_N)$, $\mathbf{f}_{-i}(\mathbf{t}_{-i})$ is the density of $\mathbf{t}_{-i}$, and $\mathbf{m}_{-i}(\mathbf{t}_{-i})$ is the message sent by $\mathbf{t}_{-i}$.

Clearly, $\tilde{V}_i(t_i) = 0$ when $t_i < t^c$. We use $u_i(\tilde{t}_i, t_i)$ to denote the expected utility of contestant i when his type is t_i and adopts the strategy $\pi_i(\tilde{t}_i)$. Given other contestants truthfully report their messages, contestant i's expected payoff when adopting the strategy $\pi_i(\tilde{t}_i)$ is

$$u_i(\tilde{t}_i, t_i) = \tilde{V}_i(\tilde{t}_i) - \frac{\int_{\mathbf{t}_{-i}} e_i(m_i(\tilde{t}_i), \mathbf{m}_{-i}(\mathbf{t}_{-i}))\mathbf{f}_{-i}(\mathbf{t}_{-i})d\mathbf{t}_{-i}}{t_i}.$$

IC and IR require that

$$u_i(t_i, t_i) \geq u_i(\tilde{t}_i, t_i), \forall t_i, \tilde{t}_i \in [t^c, b], \forall i; \tag{11}$$

$$u_i(t_i, t_i) \geq c, \forall t_i \in [t^c, b], \forall i; \tag{12}$$

$$u_i(\tilde{t}_i, t_i) \leq c, \ \forall t_i < t^c, \forall \tilde{t}_i \geq t^c, \forall i. \tag{13}$$

(11) and (13) are the incentive compatibility (IC) conditions. (12) is the entrant's participation constraint.

Define $\tilde{u}_i(\tilde{t}_i, t_i) = t_i \cdot u_i(\tilde{t}_i, t_i)$. For contestant i with type $t_i \geq t^c$, if he mimics the strategy of $\tilde{t}_i \geq t^c$, we have

$$u_i(\tilde{t}_i, t_i) = \tilde{V}_i(\tilde{t}_i) - \frac{\int_{\mathbf{t}_{-i}} e_i(m_i(\tilde{t}_i), \mathbf{m}_{-i}(\mathbf{t}_{-i}))\mathbf{f}_{-i}(\mathbf{t}_{-i})d\mathbf{t}_{-i}}{t_i}.$$

Then

$$\tilde{u}_i(\tilde{t}_i, t_i) = t_i\tilde{V}_i(\tilde{t}_i) - \int_{\mathbf{t}_{-i}} e_i(m_i(\tilde{t}_i), \mathbf{m}_{-i}(\mathbf{t}_{-i}))\mathbf{f}_{-i}(\mathbf{t}_{-i})d\mathbf{t}_{-i}.$$

By envelope theorem, we have

$$\frac{d\tilde{u}_i(t_i, t_i)}{dt_i} = \frac{\partial \tilde{u}_i(\tilde{t}_i, t_i)}{\partial t_i}|_{\tilde{t}_i = t_i} = \tilde{V}_i(t_i),$$

which leads to

$$\tilde{u}_i(t_i, t_i) - \tilde{u}_i(t^c, t^c) = \int_{t^c}^{t_i} \tilde{V}_i(s)ds, \forall t_i \geq t^c, \forall i.$$

Standard arguments as in Myerson [14] lead to that IC constraint (11) implies that

$$\int_{\mathbf{t}_{-i}} e_i(m_i(t_i),\mathbf{m}_{-i}(\mathbf{t}_{-i}))\mathbf{f}_{-i}(\mathbf{t}_{-i})d\mathbf{t}_{-i} = t_i\tilde{V}_i(t_i) - \int_{t^c}^{t_i} \tilde{V}_i(s)ds - t^c u_i(t^c,t^c), \forall t_i \geq t^c, \forall i. \tag{14}$$

Note that

$$v_i(\mathbf{s}) = e_i(\mathbf{s}) = 0, \forall \mathbf{s} \text{ with } s_i = \emptyset, \forall i,$$

since the nonparticipants does not exert effort and does not obtain any prize. Therefore, according to (14) the expected total effort inducible under the prize allocation vector $\mathbf{W}$ can be rewritten as

$$\begin{aligned} TE(\mathbf{W},t^c) &= \int_{\mathbf{t}} [\sum_i e_i(\mathbf{m}(\mathbf{t}))]\mathbf{f}(\mathbf{t})d\mathbf{t} \\ &= \sum_{i=1}^N \int_a^b \int_{\mathbf{t}_{-i}} e_i(\mathbf{m}(\mathbf{t}))f(t_i)\mathbf{f}_{-i}(\mathbf{t}_{-i})d\mathbf{t} \\ &= \sum_{i=1}^N \int_{t^c}^b \int_{\mathbf{t}_{-i}} e_i(m_i(t_i),\mathbf{m}_{-i}(\mathbf{t}_{-i}))f(t_i)\mathbf{f}_{-i}(\mathbf{t}_{-i})d\mathbf{t} \\ &= \sum_{i=1}^N \int_{t^c}^b J(t_i)\tilde{V}_i(t_i)f(t_i)dt_i - \sum_{i=1}^N u_i(t^c,t^c)t^c(1-F(t^c)). \end{aligned} \tag{15}$$

Now we are ready to prove Proposition 1 for the case of strictly increasing bidding function. Since the effort bidding function $e(t,\mathbf{W},t^c)$ must satisfy the IC constraint in the semidirect mechanism, according to (14), we have[32]

$$e(t,\mathbf{W},t^c) = t\tilde{V}(t) - \int_{t^c}^{t_i} \tilde{V}(s)ds - t^c u(t^c,t^c). \tag{16}$$

Thus, we only need to pin down the expected prize. To this end, note that since the bidding function is strictly increasing, for an entrant who mimics type $t \geq t^c$, the probability that he ranks the $n+1-j$'th place winner (i.e., there are $n-j$ entrants' types are higher than his and $j-1$ entrants' types are lower than his) is $\binom{n-1}{j-1}G^{j-1}(t,t^c)(1-G(t,t^c))^{n-j}$ if he is in scenario n, where $1 \leq j \leq n$. Moreover, the probability that scenario n happens from an entrant's perspective is $p_n(t^c) = \binom{N-1}{n-1}F^{N-n}(t^c)(1-F(t^c))^{n-1}$. Thus, the expected prize for reporting type t is

$$\tilde{V}(t) = \sum_{n=1}^N p_n(t^c) \sum_{j=1}^n w_{n,n+1-j} \binom{n-1}{j-1} G^{j-1}(t,t^c)(1-G(t,t^c))^{n-j}. \tag{17}$$

[32] Here we drop the subscript i since we focus on symmetric equilibrium.

By Lemma 4, the threshold type t^c bids zero so that

$$u(t^c, t^c) = \sum_{n=1}^{N} p_n(t^c) w_{n,n}. \tag{18}$$

Substituting (17) and (18) into (16), we obtain the expression of $e(t, \mathbf{W}, t^c)$ as in the proposition. Note that the uniqueness of the bidding function follows from the uniqueness of expected prize (17). Finally, substituting (17) into (15) and notice that $(1 - F(t^c))g(t, t^c) = f(t)$ and that $n\binom{N}{n} = N\binom{N-1}{n-1}$, we obtain the expression of $TE(\mathbf{W}, t^c)$ as in the proposition. $\square$

Proof of Lemma 6: Denote the CDF of the i'th order statistics of the n i.i.d. random variables following CDF $G(t, t^c)$ as $G_{(i,n)}(t, t^c)$. It is well know that the CDF of the i'th order statistics is $G_{(i,n)}(t, t^c) = \sum_{j=i}^{n} \binom{n}{j} G^j(t, t^c)(1 - G(t, t^c))^{n-j}$, with density $g_{(i,n)}(t, t^c) = n\binom{n-1}{i-1} G^{i-1}(t, t^c)(1 - G(t, t^c))^{n-i} g(t, t^c)$. By Lemma 5,

$$\begin{aligned} TE^{(n)}(\mathbf{W}_n, t^c) &= n \int_{t^c}^{b} J(t) V^{(n)}(t) g(t, t^c) dt - n t^c w_{n,n} \\ &= n \int_{t^c}^{b} J(t) [\sum_{j=1}^{n} w_{n,n+1-j} \binom{n-1}{j-1} G^{j-1}(t, t^c)(1 - G(t, t^c))^{n-j}] g(t, t^c) dt - n t^c w_{n,n} \\ &= \sum_{j=1}^{n} w_{n,n+1-j} \int_{t^c}^{b} J(t) g_{(j,n)}(t, t^c) dt - n t^c w_{n,n}. \end{aligned} \tag{19}$$

We first prove the following lemma.

Lemma 8 *$\forall t^c < b$, $\int_{t^c}^{b} J(t) g_{(n,n)}(t, t^c) dt > \int_{t^c}^{b} J(t) g_{(i,n)}(t, t^c) dt$ for any i with $1 \leq i \leq n-1$. Moreover, $\int_{t^c}^{b} J(t) g_{(n,n)}(t, t^c) dt > 0$.*

Proof of Lemma 8: First recall that $J(t) = t - \frac{1-F(t)}{f(t)}$ and that $G(t, t^c) = \frac{F(t) - F(t^c)}{1 - F(t^c)}, t \in [t^c, b]$.

Therefore, $J(t) = t - \frac{1-G(t,t^c)}{g(t,t^c)}, t \in [t^c, b]$. For any $1 \le i \le n$,

$$\begin{aligned}
& \int_{t^c}^{b} J(t) g_{(i,n)}(t,t^c) dt \\
= & \int_{t^c}^{b} (t g_{(i,n)}(t,t^c) - g_{(i,n)}(t,t^c) \cdot \frac{1-G(t,t^c)}{g(t,t^c)}) dt \\
= & \int_{t^c}^{b} t dG_{(i,n)}(t,t^c) - \int_{t^c}^{b} [n \binom{n-1}{i-1} G^{i-1}(t,t^c)(1-G(t,t^c))^{n-i} g(t,t^c)] \cdot \frac{1-G(t,t^c)}{g(t,t^c)}) dt \\
= & \; t G_{(i,n)}(t,t^c)|_{t=t^c}^{b} - \int_{t^c}^{b} G_{(i,n)}(t,t^c) dt - \int_{t^c}^{b} \binom{n}{i-1} (n-i+1) G^{i-1}(t,t^c)(1-G(t,t^c))^{n-i+1} dt \\
= & \; b - \int_{t^c}^{b} \sum_{j=i-1}^{n} \binom{n}{j} G^j(t,t^c)(1-G(t,t^c))^{n-j} dt - (n-i) \int_{t^c}^{b} G^{i-1}(t,t^c)(1-G(t,t^c))^{n-i+1} dt \\
\le & \; b - \int_{t^c}^{b} \sum_{j=i-1}^{n} \binom{n}{j} G^j(t,t^c)(1-G(t,t^c))^{n-j} dt \\
\le & \; b - \int_{t^c}^{b} \sum_{j=n-1}^{n} \binom{n}{j} G^j(t,t^c)(1-G(t,t^c))^{n-j} dt = \int_{t^c}^{b} J(t) g_{(n,n)}(t,t^c) dt,
\end{aligned}$$

where the last inequality is strict when $i < n$. Also notice that

$$\begin{aligned}
\int_{t^c}^{b} J(t) g_{(n,n)}(t,t^c) dt &= b - \int_{t^c}^{b} \sum_{j=n-1}^{n} \binom{n}{j} G^j(t,t^c)(1-G(t,t^c))^{n-j} dt \\
&\ge b - \int_{t^c}^{b} \sum_{j=0}^{n} \binom{n}{j} G^j(t,t^c)(1-G(t,t^c))^{n-j} dt \\
&= t^c > 0.
\end{aligned}$$

This completes the proof of the lemma. □

The above lemma actually says that the weighted average of the virtual efficiency function obtains its maximum when taking expectation with respect to the highest order statistics. Moreover, taking expectation with respect to the highest order statistics is strictly positive. Now coming back to (19). The expected total effort can be seen as a linear combination of the n prizes $w_{n,1}, w_{n,2}, \ldots, w_{n,n}$, with the coefficient $\int_{t^c}^{b} J(t) g_{(n+1-j,n)}(t,t^c) dt$ decreasing in j, minus $nt^c w_{n,n}$. Constrained by

$$\sum_{j=1}^{n} w_{n,j} = V_n, \text{ with } w_{n,n} = K_n \le \frac{V_n}{n},$$

it is obvious that setting $w_{n,1} = V_n - (n-1)K_n$ and $w_{n,j} = K_n$ for all $n \geq 2$ is optimal. $\square$

Derivation of $TE(\mathbf{V}, \mathbf{K}, t^c)$: When $\mathbf{W}_n^* = (V_n - (n-1)K_n, K_n, \ldots, K_n)$,

$$\begin{aligned}
TE^{(n)}(\mathbf{W}_n^*, t^c) &= \sum_{j=1}^{n} w_{n,n+1-j} \int_{t^c}^{b} J(t) g_{(j,n)}(t, t^c) dt - nt^c K_n \\
&= n \int_{t^c}^{b} J(t) \{[V_n - (n-1)K_n] G^{n-1}(t, t^c) + K_n [1 - G^{n-1}(t, t^c)]\} g(t, t^c) dt - nt^c K_n \\
&= n \int_{t^c}^{b} J(t)(V_n - nK_n) G^{n-1}(t, t^c) g(t, t^c) dt + nK_n \int_{t^c}^{b} J(t) g(t, t^c) dt - nt^c K_n \\
&= n \int_{t^c}^{b} J(t)(V_n - nK_n) G^{n-1}(t, t^c) g(t, t^c) dt.
\end{aligned}$$

Therefore, the expected total effort

$$\begin{aligned}
TE(\mathbf{V}, \mathbf{K}, t^c) &= \sum_{n=1}^{N} \binom{N}{n} (1 - F(t^c))^n F^{N-n}(t^c) TE^{(n)}(\mathbf{W}_n^*, t^c) \\
&= \sum_{n=1}^{N} \binom{N}{n} (1 - F(t^c))^n F^{N-n}(t^c) n \int_{t^c}^{b} J(t)(V_n - nK_n) G^{n-1}(t, t^c) g(t, t^c) dt \\
&= N(1 - F(t^c)) \sum_{n=1}^{N} \binom{N-1}{n-1} (1 - F(t^c))^{n-1} F^{N-n}(t^c) \\
&\quad \cdot \int_{t^c}^{b} J(t)(V_n - nK_n) G^{n-1}(t, t^c) g(t, t^c) dt \\
&= N(1 - F(t^c)) \sum_{n=1}^{N} p_n(t^c) \int_{t^c}^{b} J(t)(V_n - nK_n) G^{n-1}(t, t^c) g(t, t^c) dt.
\end{aligned}$$

$\square$

Proof of Proposition 2: Let $\mathbf{K} = (K_1, K_2, \ldots, K_N)$ be the optimal minimum prize vector. Note that as discussed before Proposition 2, there is no loss to assume that $V_n = 1$ for all $n \geq 2$. And it is obvious that $V_1 = K_1$. Since $\mathbf{K}$ satisfies (6), we have $\sum_{j=1}^{N} p_j(t^c) K_j = c$. If $t^c = a$, then $p_n(t^c) = 0$ for all $n \leq N-1$, and $t^c \in [t_N, t_{N-1}]$. In this case, only K_N is relevant. It is obvious that from (6) $K_N = c$, which satisfies the expression given in Proposition 2. Note that as mentioned in footnote 26, the optimal K_n for $n \leq N-1$ may not be unique as they do not enter the objective function.

Now we come to the case that $t^c > a$. First notice that $\mathbf{V}^*(t^c)$ and $\mathbf{K}^*(t^c)$ given in Proposition 2 satisfy all the constraints. First consider the case that $t^c \in [t_1, b)$. If there is some $m \geq 2$ such that $K_m > 0$, then $K_1 < K_1^*(t^c) = \frac{c}{F^{N-1}(t^c)}$ by (6). Next consider the case that $t^c \in [t_n, t_{n-1}]$ for some $n \geq 2$. If in $\mathbf{K}$ there is some $m \geq n+1$ such that $K_m > 0$, then there must exist some $m' \leq n$ such that $K_{m'} < K_{m'}^*(t^c)$. To see this, note that if for all $m' \leq n$, $K_{m'} = K_{m'}^*(t^c) = \frac{1}{m'}$, then

$$c = \sum_{j=1}^{N} p_j(t^c)K_j \geq p_m(t^c)K_m + \sum_{j=1}^{n} \frac{p_j(t^c)}{j} > \sum_{j=1}^{n} \frac{p_j(t^c)}{j} \geq \sum_{j=1}^{n} \frac{p_j(t_n)}{j} = c.$$

The first equality is from (6), the second inequality is due to $K_m > 0$, the third inequality is because $\sum_{j=1}^{n} \frac{p_j(t^c)}{j}$ is strictly increasing in $t^c \in (a, b)$,[33] and the last equality comes from the definition of t_n. A contradiction.

Therefore, if $t^c \in [t_n, t_{n-1}]$ for some $n \geq 1$, and if there is some $m \geq n+1$ such that $K_m > 0$, then there must exist some $m' \leq n$, such that $K_{m'} < K_{m'}^*(t^c)$. Now consider the following new construction of minimum prize vector.[34] Let $\mathbf{K}' = (K_1, \ldots, K_{m'-1}, K'_{m'}, K_{m'+1}, \ldots, K_{m-1}, K'_m, K_{m+1}, \ldots, K_N)$. That is, change $K_{m'}$ and K_m to $K'_{m'}$ and K'_m, respectively, to form $\mathbf{K}'$. Choose $\varepsilon > 0$ such that $K'_{m'} = K_{m'} + \varepsilon \leq K_{m'}^*(t^c)$. To maintain the equality in (6), K_m should be changed to $K'_m = K_m - \frac{p_{m'}(t^c)}{p_m(t^c)}\varepsilon \geq 0$.[35] The change of total effort is $TE(t^c; \mathbf{K}') - TE(t^c; \mathbf{K})$. Our goal is to show that this difference is strictly positive.

To this end, notice that the difference can be expressed as (dropping the common factor $N(1 - F(t^c))$)

$$\begin{aligned} & p_m(t^c) \int_{t^c}^{b} J(t) m (\frac{p_{m'}(t^c)}{p_m(t^c)}\varepsilon) G^{m-1}(t, t^c) g(t, t^c) dt - p_{m'}(t^c) \int_{t^c}^{b} J(t) m' \varepsilon G^{m'-1}(t, t^c) g(t, t^c) dt \\ = \; & p_{m'}(t^c)\varepsilon \int_{t^c}^{b} J(t)[mG^{m-1}(t, t^c) - m'G^{m'-1}(t, t^c)] g(t, t^c) dt. \end{aligned}$$

We need to show that

$$\int_{t^c}^{b} J(t)[mG^{m-1}(t, t^c) - m'G^{m'-1}(t, t^c)] g(t, t^c) dt > 0 \text{ when } m > m' \geq 1. \tag{20}$$

It suffices to show that, for any $k \geq 1$,

$$\int_{t^c}^{b} J(t) k G^{k-1}(t, t^c) g(t, t^c) dt < \int_{t^c}^{b} J(t)(k+1) G^{k}(t, t^c) g(t, t^c) dt. \tag{21}$$

[33] See footnote 24.

[34] Note that there is no loss to assume that the budget $V_n = 1$ for $n \geq 2$. The budget for $n = 1$ must be equal to the minimum prize K_1.

[35] Note that since $p_k(t^c) \neq 0$ for any $t^c \in (a, b)$ and any k. Thus, $K'_m < K_m$. Also note that $K'_m \geq 0$ is always feasible since we can choose $\varepsilon > 0$ small enough.

In fact,

$$\begin{aligned}
&\int_{t^c}^{b} J(t)kG^{k-1}(t,t^c)g(t,t^c)dt \\
= \;& k\int_{t^c}^{b}(t-\frac{1-G(t,t^c)}{g(t,t^c)})G^{k-1}(t,t^c)g(t,t^c)dt \\
= \;& \int_{t^c}^{b} tdG^k(t,t^c) - k\int_{t^c}^{b} G^{k-1}(t,t^c)(1-G(t,t^c))dt \\
= \;& tG^k(t,t^c)|_{t=t^c}^{b} - \int_{t^c}^{b} G^k(t,t^c)dt - k\int_{t^c}^{b} G^{k-1}(t,t^c)(1-G(t,t^c))dt \\
= \;& b - k\int_{t^c}^{b} G^{k-1}(t,t^c)dt + (k-1)\int_{t^c}^{b} G^k(t,t^c)dt.
\end{aligned}$$

Thus, showing (21) is equivalent to proving

$$-k\int_{t^c}^{b} G^{k-1}(t,t^c)dt + (k-1)\int_{t^c}^{b} G^k(t,t^c)dt < -(k+1)\int_{t^c}^{b} G^k(t,t^c)dt + k\int_{t^c}^{b} G^{k+1}(t,t^c)dt.$$

That is,

$$k\int_{t^c}^{b} G^{k-1}(t,t^c)(1-G(t,t^c))^2dt > 0,$$

which obviously holds.

As a result, $\int_{t^c}^{b} J(t)kG^{k-1}(t,t^c)g(t,t^c)dt$ is strictly increasing in k when $k \geq 1$. Therefore, (20) is satisfied.

The above argument shows that if the candidate $\mathbf{K} = (K_1, K_2, \ldots, K_N)$ satisfies that there is some $m \geq n+1$ such that $K_m > 0$, where n is the number such that $t^c \in [t_n, t_{n-1}]$, then such $\mathbf{K}$ can never be optimal. As a result, the optimal $\mathbf{K}$ must satisfy $K_m = 0$ for all $m \geq n+1$. If $n = 1$, then it means $K_n = 0$ for all $n \geq 2$ so that (6) implies that $K_1 = \frac{c}{F^{N-1}(t^c)}$. If $n \geq 2$, then it must be the case that $K_m = \frac{1}{m}$ for all $m \leq n-1$. To see this, suppose to the contrary that there is some $m \leq n-1$ such that $K_m < \frac{1}{m}$. Then $K_n > 0$ by the definition of t_n and the fact that $\sum_{j=1}^{n} \frac{p_j(t^c)}{j}$ is strictly increasing in t^c and that $K_j \leq \frac{1}{j}$ for all j. This implies that we can decrease K_n to $K_n' \geq 0$ and increase K_m to $K_m' \leq \frac{1}{m}$ by proper amounts just as what we did in the argument of proving $K_j = 0$ for all $j \geq n+1$. Similar argument leads to that the expected total effort increase after such change. Therefore, when $n \geq 2$, we have $K_m = \frac{1}{m}$ for all $m \leq n-1$. The value of K_n is determined by (6) by setting $K_m = \frac{1}{m}$ for all $m \leq n-1$ and $K_m = 0$ for all $m \geq n+1$. This completes the proof. □

Proof of Lemma 7: We first introduce and recall some notations. Recall that $G(t,t^c) = \frac{F(t)-F(t^c)}{1-F(t^c)}$, $g(t,t^c) = \frac{f(t)}{1-F(t^c)}$. Then $\frac{dG(t,t^c)}{dt^c} = -\frac{f(t^c)(1-G(t,t^c))}{1-F(t^c)}$. Define $\varphi(n,t^c) = n\int_{t^c}^{b} J(t)G^{n-1}(t,t^c)g(t,t^c)dt$,

$n \geq 1$. By the proof of Proposition 2, $\varphi(n+1, t^c) > \varphi(n, t^c) > \ldots > \varphi(1, t^c) = t^c > 0$, $\forall t^c < b$. From now on, we will neglect the common factor N in the expression of $TE^*(t^c)$ as it does not affect the monotonicity. By a little abuse of notation, we also use $TE^*(t^c)$ to denote it.

We first show that $TE^*(t^c)$ is strictly decreasing in $[t_1, b)$. According to Proposition 2,

$$\begin{aligned} TE^*(t^c) &= (1 - F(t^c)) \sum_{n=2}^{N} p_n(t^c) \int_{t^c}^{b} J(t) G^{n-1}(t, t^c) g(t, t^c) dt \\ &= \sum_{n=2}^{N} \binom{N-1}{n-1} F^{N-n}(t^c) \int_{t^c}^{b} J(t) [F(t) - F(t^c)]^{n-1} f(t) dt \\ &= \int_{t^c}^{b} J(t) [F^{N-1}(t) - F^{N-1}(t^c)] f(t) dt. \end{aligned}$$

Thus,

$$\begin{aligned} \frac{dTE^*}{dt^c} &= -(N-1) f(t^c) \int_{t^c}^{b} J(t) F^{N-2}(t^c) f(t) dt \\ &= -(N-1) F^{N-2}(t^c) f(t^c) t^c (1 - F(t^c)) < 0. \end{aligned}$$

That is, $TE^*(t^c)$ is strictly decreasing in $[t_1, b)$.

Now we go on to show that $TE^*(t^c)$ is strictly increasing in $[a, t_1]$. It suffices to show that it is strictly increasing in $[t_n, t_{n-1}]$ for every $n \geq 2$. When $t^c \in [t_n, t_{n-1}]$, according to Proposition 2,

$$\begin{aligned} TE^*(t^c) &= (1 - F(t^c)) \sum_{j=1}^{N} p_j(t^c) \int_{t^c}^{b} J(t)(1 - jK_j^*(t^c)) G^{j-1}(t, t^c) g(t, t^c) dt \\ &= (1 - F(t^c)) \sum_{j=n}^{N} \binom{N-1}{j-1} (1 - F(t^c))^{j-1} F^{N-j}(t^c) \int_{t^c}^{b} J(t) G^{j-1}(t, t^c) g(t, t^c) dt \\ &\quad - nK_n^*(t^c) p_n(t^c)(1 - F(t^c)) \int_{t^c}^{b} J(t) G^{n-1}(t, t^c) g(t, t^c) dt \\ &= \underbrace{\sum_{j=n}^{N} \binom{N-1}{j-1} F^{N-j}(t^c) \int_{t^c}^{b} J(t) [F(t) - F(t^c)]^{j-1} f(t) dt}_{A(t^c)} \\ &\quad \underbrace{- n(c - \sum_{j=1}^{n-1} \frac{p_j(t^c)}{j}) \int_{t^c}^{b} J(t) G^{n-1}(t, t^c) f(t) dt.}_{B(t^c)} \end{aligned}$$

Note that

$$A(t^c) = \int_{t^c}^{b} J(t)f(t)\underbrace{\{F^{N-1}(t) - \sum_{j=1}^{n-1}\binom{N-1}{j-1}[F(t)-F(t^c)]^{j-1}F^{N-j}(t^c)\}}_{A_1(t,t^c)}dt.$$

We have[36]

$$\begin{aligned}
\frac{dA_1}{dt^c} &= -f(t^c)\{-\sum_{j=1}^{n-1}\binom{N-1}{j-1}(j-1)[F(t)-F(t^c)]^{j-2}F^{N-j}(t^c) \\
&\quad +\sum_{j=1}^{n-1}\binom{N-1}{j-1}(N-j)[F(t)-F(t^c)]^{j-1}F^{N-j-1}(t^c)\} \\
&= (N-1)f(t^c)\{\begin{array}{c}\sum_{j=2}^{n-1}\binom{N-2}{j-2}[F(t)-F(t^c)]^{j-2}F^{N-j}(t^c) \\ -\sum_{j=1}^{n-1}\binom{N-2}{j-1}[F(t)-F(t^c)]^{j-1}F^{N-j-1}(t^c)\end{array}\} \\
&= (N-1)f(t^c)\{\begin{array}{c}\sum_{j=2}^{n-1}\binom{N-2}{j-2}[F(t)-F(t^c)]^{j-2}F^{N-j}(t^c) \\ -\sum_{k=2}^{n}\binom{N-2}{k-2}[F(t)-F(t^c)]^{k-2}F^{N-k}(t^c)\end{array}\} \\
&= -(N-1)f(t^c)\binom{N-2}{n-2}[F(t)-F(t^c)]^{n-2}F^{N-n}(t^c).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\frac{dA}{dt^c} &= -J(t^c)f(t^c)A_1(t^c,t^c) - (N-1)f(t^c)\binom{N-2}{n-2}F^{N-n}(t^c)\int_{t^c}^{b} J(t)[F(t)-F(t^c)]^{n-2}f(t)dt \\
&= -f(t^c)\frac{N-1}{n-1}\binom{N-2}{n-2}F^{N-n}(t^c)(n-1)\int_{t^c}^{b} J(t)[F(t)-F(t^c)]^{n-2}f(t)dt \\
&= -f(t^c)\binom{N-1}{n-1}(1-F(t^c))^{n-1}F^{N-n}(t^c)(n-1)\int_{t^c}^{b} J(t)G^{n-2}(t,t^c)g(t,t^c)dt \\
&= -p_n(t^c)f(t^c)\varphi(n-1,t^c).
\end{aligned}$$

[36] Define $\sum_{j=m}^{m'}\phi(j) = 0$ when $m > m'$, where $\phi(\cdot)$ is any function.

Now we turn to $B(t^c)$. We first look at the derivative of $\sum_{j=1}^{n-1} \frac{p_j(t^c)}{j}$ with respect to t^c.

$$\frac{d}{dt^c}\sum_{j=1}^{n-1}\frac{1}{j}\binom{N-1}{j-1}(1-F(t^c))^{j-1}F^{N-j}(t^c)$$

$$= \begin{array}{c} -f(t^c)\sum_{j=1}^{n-1}\binom{N-1}{j-1}\frac{j-1}{j}(1-F(t^c))^{j-2}F^{N-j}(t^c) \\ +f(t^c)\sum_{j=1}^{n-1}\binom{N-1}{j-1}\frac{N-j}{j}(1-F(t^c))^{j-1}F^{N-j-1}(t^c) \end{array}$$

$$= (N-1)f(t^c)[\begin{array}{c} \sum_{j=1}^{n-1}\frac{1}{j}\binom{N-2}{j-1}(1-F(t^c))^{j-1}F^{N-j-1}(t^c) \\ -\sum_{j=2}^{n-1}\frac{1}{j}\binom{N-2}{j-2}(1-F(t^c))^{j-2}F^{N-j}(t^c) \end{array}]$$

$$= (N-1)f(t^c)[\begin{array}{c} \sum_{k=2}^{n}\frac{1}{k-1}\binom{N-2}{k-2}(1-F(t^c))^{k-2}F^{N-k}(t^c) \\ -\sum_{j=2}^{n-1}\frac{1}{j}\binom{N-2}{j-2}(1-F(t^c))^{j-2}F^{N-j}(t^c) \end{array}]$$

$$= (N-1)f(t^c)[\begin{array}{c} \frac{1}{n-1}\binom{N-2}{n-2}(1-F(t^c))^{n-2}F^{N-n}(t^c) \\ +\sum_{j=2}^{n-1}\frac{1}{j(j-1)}\binom{N-2}{j-2}(1-F(t^c))^{j-2}F^{N-j}(t^c) \end{array}]$$

$$= \frac{f(t^c)}{1-F(t^c)}(p_n(t^c)+\sum_{j=2}^{n-1}\frac{p_j(t^c)}{j}).$$

Thus,

$$\begin{aligned} \frac{dB}{dt^c} &= -n(\frac{d}{dt^c}\sum_{j=1}^{n-1}\frac{p_j(t^c)}{j})\int_{t^c}^{b}J(t)G^{n-1}(t,t^c)f(t)dt - \\ & \frac{f(t^c)}{1-F(t^c)}n(c-\sum_{j=1}^{n-1}\frac{p_j(t^c)}{j})(n-1)\int_{t^c}^{b}J(t)G^{n-2}(t,t^c)(1-G(t,t^c))f(t)dt \\ &= -\frac{nf(t^c)}{1-F(t^c)}(p_n(t^c)+\sum_{j=2}^{n-1}\frac{p_j(t^c)}{j})\int_{t^c}^{b}J(t)G^{n-1}(t,t^c)f(t)dt \\ & -n(n-1)(c-\sum_{j=1}^{n-1}\frac{p_j(t^c)}{j})f(t^c)\int_{t^c}^{b}J(t)(G^{n-2}(t,t^c)-G^{n-1}(t,t^c))g(t,t^c)dt \\ &= -f(t^c)\{\begin{array}{c} (p_n(t^c)+\sum_{j=2}^{n-1}\frac{p_j(t^c)}{j})\varphi(n,t^c) \\ +(c-\sum_{j=1}^{n-1}\frac{p_j(t^c)}{j})[n\varphi(n-1,t^c)-(n-1)\varphi(n,t^c)] \end{array}\}. \end{aligned}$$

Therefore,

$$\begin{aligned}
\frac{dTE^*}{dt^c} &= \frac{dA}{dt^c} - \frac{dB}{dt^c} \\
&= f(t^c)\{-p_n(t^c)\varphi(n-1,t^c) + (p_n(t^c) + \sum_{j=2}^{n-1}\frac{p_j(t^c)}{j})\varphi(n,t^c) \\
&\quad +(c - \sum_{j=1}^{n-1}\frac{p_j(t^c)}{j})[n\varphi(n-1,t^c) - (n-1)\varphi(n,t^c)]\} \\
&= f(t^c)[(\beta(t^c) + c - p_1(t^c))\varphi(n,t^c) - \beta(t^c)\varphi(n-1,t^c)],
\end{aligned}$$

where $\beta(t^c) = p_n(t^c) + n\sum_{j=1}^{n-1}\frac{p_j(t^c)}{j} - nc \geq 0$ by the definition of t_n and the fact that $\sum_{j=1}^{n}\frac{p_j(t^c)}{j}$ is increasing in t^c. Also notice that $c - p_1(t^c) > 0$ when $t^c < t_1$. Therefore, when $t^c < t_1$,

$$\begin{aligned}
\frac{dTE^*}{dt^c} &= f(t^c)[\beta(t^c)(\varphi(n,t^c) - \varphi(n-1,t^c)) + (c - p_1(t^c))\varphi(n,t^c)] \\
&\geq (c - p_1(t^c))\varphi(n,t^c)f(t^c) > 0.
\end{aligned}$$

The first inequality follows from $\beta(t^c) \geq 0$ and $\varphi(n,t^c) > \varphi(n-1,t^c)$ when $n \geq 2$. The second inequality follows from $c > p_1(t^c)$ and $\varphi(n,t^c) > 0$ when $n \geq 2$.

Thus, we have shown that $TE^*(t^c)$ is strictly increasing in $[a, t_1]$. This completes the proof of Lemma 7. □

Proof of Remark 3: When $c \geq 1$, it is obvious that no entrant will enter since the organizer's budget cannot cover the entrant's entry cost. If $c \in (1/N, 1)$, then it means that there exists some cutoff type $\tilde{t}$ such that all types lower than it cannot be supported as entry thresholds. To see this, notice that for any given entry threshold t^c, the highest payoff the type t^c entrant can have is $\sum_{j=1}^{N}\frac{p_j(t^c)}{j}$. If $\sum_{j=1}^{N}\frac{p_j(t^c)}{j} < c$, then t^c cannot be supported as an entry threshold. Recall that $\sum_{j=1}^{n}\frac{p_j(t^c)}{j}$ is strictly increasing in $t^c \in (a,b)$ for any $n \geq 1$. Also notice that $\sum_{j=1}^{N}\frac{p_j(a)}{j} = \frac{1}{N} < c$, and $\sum_{j=1}^{N}\frac{p_j(b)}{j} = 1 > c$. Therefore, there exists a unique $\tilde{t}_N \in (a,b)$ such that $\sum_{j=1}^{N}\frac{p_j(\tilde{t}_N)}{j} = c$. Then any type which is strictly smaller than $\tilde{t}_N$ cannot be supported as an entry threshold. Thus, an entry threshold is inducible if and only if it is weakly higher than $\tilde{t}_N$ so that there is no loss of generality to confine attention to thresholds in the interval $[\tilde{t}_N, b)$. Similarly to the construction of t_n, let $\tilde{t}_n$ be the unique solution to equation (7), where $n < N$. (In fact, $\tilde{t}_n = t_n$.) Then similar arguments as those in the proof of Lemma 7 imply that setting the entry threshold as t_1 is optimal. Therefore, when $c \in (1/N, 1)$, winner-take-all with prize 1 is still optimal. □

6.2 Proofs of the Concave Cost Case

Denote $h = \tilde{h}^{-1}$, then $h(0) = 0$, $h' > 0$, and $h'' \geq 0$. For any given entry threshold t^c and any given budget vector $\mathbf{V}$ and $\mathbf{W}$ that support t^c,[37] it is easy to see that

$$e(t, \mathbf{W}, t^c) = h[\sum_{n=1}^{N} p_n(t^c) e^{(n)}(t, \mathbf{W}_n, t^c)], t \geq t^c,$$

where $p_n(t^c) = \binom{N-1}{n-1} F^{N-n}(t^c)(1 - F(t^c))^{n-1}$, as defined before, is the probability of scenario n happening from an entrant's perspective, and $e^{(n)}(t, \mathbf{W}_n, t^c)$ is the one given in Lemma 5. Therefore, the expected total effort is

$$\begin{aligned} TE(\mathbf{W}, t^c) &= \sum_{n=1}^{N} \binom{N}{n} (1 - F(t^c))^n F^{N-n}(t^c) n \int_{t^c}^{b} h(e(t, \mathbf{W}, t^c)) g(t, t^c) dt \\ &= N(1 - F(t^c)) \int_{t^c}^{b} h(e(t, \mathbf{W}, t^c)) g(t, t^c) dt. \end{aligned}$$

The analysis of the optimal prize allocation rule proceeds exactly as the one in the linear cost case. First we fix the entry threshold as t^c to find the optimal prize allocation rule that induces it. And then varying across all the entry thresholds will lead to the optimum. For the first step, similarly to the linear cost case, we first fix the budget vector $\mathbf{V} = (V_1, V_2, \ldots, V_N)$ and the minimum prize vector $\mathbf{K} = (K_1, K_2, \ldots, K_N)$ to find the optimal prize allocation rule in this restricted situation. The optimal prize allocation rule turns out to be exactly the same as the one characterized in Lemma 6. The reason is that: 1) The threshold type bids zero so that his payoff is the same as that in the linear case; 2) The cross-rank transfer still applies. The proof is similar to that in Moldovanu and Sela [9] except that now every prize has a lower bound of K_n in scenario n.

After characterizing the optimal prize allocation rule for fixed t^c, $\mathbf{V}$, and $\mathbf{W}$, the next step, similarly, is to find the optimal $(\mathbf{V}, \mathbf{K})$ that induces t^c. Recall that the crucial step in the linear case to derive the optimum is the cross-scenario transfer, which will be shown to hold in the concave cost case. The final step is to show the optimality of winner-take-all.

Proof of cross-rank transfer: In scenario $n \geq 2$, suppose the prize allocation rule is $\mathbf{W}_n = (w_{n,1}, w_{n,2}, \ldots, w_{n,n})$ so that the minimum prize is $K_n = w_{n,n}$. It suffices to show that, fixing the allocation rule in all other scenarios, at the optimum, it must be the case that $w_{n,n+1-j} = K_n$ for all $j < n$. Suppose to the contrary that there exists some j with $1 < j < n$ such that $w_{n,n+1-j} > K_n$.[38]

[37] In fact, note that once $\mathbf{W}$ is given, $\mathbf{V}$ is also pinned down.

[38] Recall that K_n is defined as $w_{n,n}$.

If there are more than one j's, pick the largest one. The expected total effort can be expressed as

$$TE = N(1-F(t^c))\int_{t^c}^{b} h(\sum_{k=1}^{N} p_k(t^c)e^{(k)}(t,\mathbf{W}_k,t^c))g(t,t^c)dt,$$

where $e^{(k)}(t,\mathbf{W}_n,t^c)$ is given in Lemma 5.

For notation simplicity, let $\theta(t,t^c)$ denote $\sum_{k=1}^{N} p_k(t^c)e^{(k)}(t,\mathbf{W}_k,t^c)$. We only need to show that

$$\frac{dTE}{dw_{n,1}} > \frac{dTE}{dw_{n,n+1-j}}. \tag{22}$$

If this holds, then reducing $w_{n,n+1-j}$ by ε while increasing $w_{n,1}$ by ε will increase the expected total effort.

Note that (neglect the term $N(1-F(t^c))$)

$$\begin{aligned}
\frac{dTE}{dw_{n,1}} &= p_n(t^c)\int_{t^c}^{b} h'(\theta)[tG^{n-1}(t,t^c) - \int_{t^c}^{t} G^{n-1}(s)ds]g(t,t^c)dt, \\
\frac{dTE}{dw_{n,n+1-j}} &= p_n(t^c)\int_{t^c}^{b} h'(\theta)[\begin{array}{c} t\binom{n-1}{j-1}G^{j-1}(t,t^c)(1-G(t,t^c))^{n-j} \\ -\int_{t^c}^{t}\binom{n-1}{j-1}G^{j-1}(s,t^c)(1-G(s,t^c))^{n-j}ds \end{array}]g(t,t^c)dt.
\end{aligned}$$

Denote the term in the bracket in the integrand in the first equality as $B_1(t,t^c)$, and the term in the bracket in the second integrand in the second equality as $B_j(t,t^c)$. From the analysis of the cross-rank transfer in the linear cost case (i.e., $h'=1$), we know that $\int_{t^c}^{b}(B_1(t,t^c)-B_j(t,t^c))g(t,t^c)dt > 0$. Our goal is to show that $\int_{t^c}^{b} h'(\theta)(B_1(t,t^c)-B_j(t,t^c))g(t,t^c)dt > 0$.

When $t > t^c$,

$$\frac{d(B_1-B_j)}{dt} = tG^{n-2}(t,t^c)g(t,t^c)[n-1-\binom{n-1}{j-1}(\frac{1}{G(t,t^c)}-1)^{n-j-1}(\frac{j-1}{G(t,t^c)}-n+1)].$$

Thus there exits a unique $t^* \in (t^c,b)$ such that $\frac{d(B_1-B_j)}{dt} < 0$ when $t \in (t^c,t^*)$; $\frac{d(B_1-B_j)}{dt} = 0$ when $t = t^*$; and $\frac{d(B_1-B_j)}{dt} > 0$ when $t > t^*$. Combining with the fact that $B_1(t^c,t^c) = B_j(t^c,t^c) = 0$ and $\int_{t^c}^{b}(B_1(t,t^c)-B_j(t,t^c))g(t,t^c) > 0$, there exists a unique $t^{**} \in (t^c,b)$ such that $B_1(t,t^c)-B_j(t,t^c) < 0$ when $t \in (t^c,t^{**})$; $B_1(t,t^c)-B_j(t,t^c) = 0$ when $t = t^{**}$; and $B_1(t,t^c)-B_j(t,t^c) > 0$ when $t > t^{**}$.

Note that $\frac{dh'(\theta)}{dt} = h''(\theta)\frac{d\theta}{dt} \geq 0$, since $h'' \geq 0$ and $\frac{d\theta}{dt} \geq 0$ as the bidding function in the linear cost case, which is $\theta(t,t^c)$, is weakly increasing in type. Therefore,

$$\begin{aligned}
&\int_{t^c}^{b} h'(\theta)(B_1(t,t^c)-B_j(t,t^c))g(t,t^c)dt \\
= \;& (\int_{t^c}^{t^{**}} + \int_{t^{**}}^{b})[h'(\theta)(B_1(t,t^c)-B_j(t,t^c))g(t,t^c)]dt \\
\geq \;& h'(\theta(t^*,t^c))\int_{t^c}^{b}(B_1(t,t^c)-B_j(t,t^c))g(t,t^c)dt > 0.
\end{aligned}$$

This completes the proof. □

Proof of cross-scenario transfer: Given a fixed minimum prize vector $\mathbf{K}$ and budget vector $\mathbf{V}$, the expected total effort can be rewritten as

$$TE = N(1-F(t^c))\int_{t^c}^{b} h\underbrace{\{\sum_{n=1}^{N} p_n(t^c)(V_n - nK_n)(tG^{n-1}(t,t^c) - \int_{t^c}^{t} G^{n-1}(s,t^c)ds)\}}_{\theta(t,t^c)})g(t,t^c)dt.$$

Define

$$\beta_n(t,t^c) = n(tG^{n-1}(t,t^c) - \int_{t^c}^{t} G^{n-1}(s,t^c)ds).$$

The minimum prize vector $\mathbf{K}$ satisfies the following constraint:

$$\sum_{n=1}^{N} p_n(t^c)K_n = c.$$

An $\varepsilon/p_n(t^c)$ decrease in K_n should be compensated by an $\varepsilon/p_m(t^c)$ increase in K_m $(m \neq n)$. Note that for small enough $\varepsilon > 0$, an $\varepsilon/p_n(t^c)$ decrease in K_n will increase the expected total effort by

$$N(1-F(t^c))\varepsilon\int_{t^c}^{b} h'(\theta(t,t^c))\beta_n(t,t^c)g(t,t^c)dt.$$

An $\varepsilon/p_m(t^c)$ increase in K_m will decrease the expected total effort by

$$N(1-F(t^c))\varepsilon\int_{t^c}^{b} h'(\theta(t,t^c))\beta_m(t,t^c)g(t,t^c)dt.$$

To show the cross-scenario transfer, it suffices to show that

$$\int_{t^c}^{b} h'(\theta)\beta_n(t,t^c)g(t,t^c)dt > \int_{t^c}^{b} h'(\theta)\beta_{n-1}(t,t^c)g(t,t^c)dt, \forall n \geq 2. \tag{23}$$

Note that in the linear cost case which corresponds to $h' = 1$, we have shown that

$$\int_{t^c}^{b} \beta_n(t,t^c)g(t,t^c)dt > \int_{t^c}^{b} \beta_{n-1}(t,t^c)g(t,t^c)dt. \tag{24}$$

Notice that $\beta_1(t^c) = t^c > 0$, and when $n \geq 3$, $\beta_n(t^c,t^c) = \beta_{n-1}(t^c,t^c) = 0$; $\frac{d\beta_n(t,t^c)}{dt} < \frac{d\beta_{n-1}(t,t^c)}{dt}$ when $t < t^*$; $\frac{d\beta_n(t,t^c)}{dt} = \frac{d\beta_{n-1}(t,t^c)}{dt}$ when $t = t^*$; $\frac{d\beta_n(t,t^c)}{dt} > \frac{d\beta_{n-1}(t,t^c)}{dt}$ when $t > t^*$. Here $G(t^*,t^c) = 1 - \frac{2}{n}$. According to (24), we must have $\beta_n(b,t^c) > \beta_{n-1}(b,t^c)$. Thus, there exists a unique $t^{**} \in (t^c, b)$ such that $\beta_n(t,t^c) - \beta_{n-1}(t,t^c) < 0$ when $t \in (t^c, t^{**})$, $\beta_n(t,t^c) - \beta_{n-1}(t,t^c) = 0$ when $t = t^{**}$, and $\beta_n(t,t^c) - \beta_{n-1}(t,t^c) > 0$ when $t > t^{**}$.

Similar to the proof of cross-rank transfer in the concave cost case, we have $\frac{dh'(\theta)}{dt} = h''(\theta)\frac{d\theta}{dt} \geq 0$. Therefore,

$$\begin{aligned}
& \int_{t^c}^{b} h'(\theta)(\beta_n(t,t^c)) - \beta_{n-1}(t,t^c))g(t,t^c)dt \\
= \; & (\int_{t^c}^{t^{**}} + \int_{t^{**}}^{b})[h'(\theta)(\beta_n(t,t^c)) - \beta_{n-1}(t,t^c))]g(t,t^c)dt \\
\geq \; & h'(\theta(t^{**},t^c))(\int_{t^c}^{t^{**}} + \int_{t^{**}}^{b})(\beta_n(t,t^c) - \beta_{n-1}(t,t^c))g(t,t^c)dt \\
= \; & h'(\theta(t^{**},t^c)) \int_{t^c}^{b} (\beta_n(t,t^c) - \beta_{n-1}(t,t^c))g(t,t^c)dt > 0.
\end{aligned}$$

Therefore, (23) holds. In other words, the cross-scenario transfer property still holds in the concave cost case. □

Proof of the optimality of winner-take-all: We first show that the expected total effort is strictly decreasing in t^c when $t^c > t_1$. For notation simplicity, define

$$\gamma_n(t,t^c) = tG^{n-1}(t,t^c) - \int_{t^c}^{t} G^{n-1}(s,t^c)ds.$$

In fact, the expected total effort now can be rewritten as[39]

$$\begin{aligned}
TE^*(t^c) &= N(1-F(t^c)) \int_{t^c}^{b} h(\sum_{n=2}^{N} p_n(t^c)\gamma_n(t,t^c))g(t,t^c)dt \\
&= N \int_{t^c}^{b} h(\sum_{n=2}^{N} p_n(t^c)\gamma_n(t,t^c))f(t)dt.
\end{aligned}$$

Thus,

$$\frac{dTE^*}{dt^c} = N \int_{t^c}^{b} h'(\sum_{n=2}^{N} p_n(t^c)\gamma_n(t,t^c)) \frac{d}{dt^c}(\sum_{n=2}^{N} p_n(t^c)\gamma_n(t,t^c))f(t)dt.$$

[39] Note that similarly to the analysis in the linear cost case, the organizer will always exhaust her budget when there are at least two entrants. Also note that, in the expressions of the expected total effort in this proof, all of them are the expressions under the optimal prize allocation rule for the corresponding entry threshold.

Note that

$$\begin{aligned}
&\sum_{n=2}^{N} p_n(t^c)\gamma_n(t,t^c) \\
= \quad & \sum_{n=2}^{N} \binom{N-1}{n-1}(1-F(t^c))^{n-1}F^{N-n}(t^c)[tG^{n-1}(t,t^c) - \int_{t^c}^{t} G^{n-1}(s,t^c)ds] \\
= \quad & t[F^{N-1}(t) - F^{N-1}(t^c)] - \int_{t^c}^{t}[F^{N-1}(s) - F^{N-1}(t^c)]ds \\
= \quad & tF^{N-1}(t) - \int_{t^c}^{t} F^{N-1}(s)ds - t^c F^{N-1}(t^c).
\end{aligned}$$

Thus,

$$\frac{d}{dt^c}[\sum_{n=2}^{N} p_n(t^c)\gamma_n(t,t^c)] = -(N-1)F^{N-2}(t^c)t^c f(t^c) < 0.$$

Together with $h' > 0$, we have $\frac{dTE^*}{dt^c} < 0$.

Now we go on to show that $TE^*(t^c)$ is strictly increasing in $[a, t_1]$. Similar to the proof in the linear case, we only need to show that the expected total effort is strictly increasing in $[t_n, t_{n-1}]$ for every $n \geq 2$. When $t^c \in [t_n, t_{n-1}]$, the expected total effort can be expressed as

$$TE^*(t^c) = N\int_{t^c}^{b} h[\underbrace{\sum_{k=n}^{N} p_k(t^c)\gamma_k(t,t^c) - nK_n^*(t^c)p_n(t^c)\gamma_n(t,t^c)}_{\tilde{\theta}(t,t^c)}]f(t)dt,$$

where $\gamma_k(t,t^c) = tG^{k-1}(t,t^c) - \int_{t^c}^{t} G^{k-1}(s,t^c)ds$, as defined before. Then

$$\begin{aligned}
\frac{dTE^*}{dt^c} &= -Nh(\tilde{\theta}(t^c,t^c))f(t^c) + N\int_{t^c}^{b} h'(\tilde{\theta})\frac{\partial\tilde{\theta}}{\partial t^c}f(t)dt \\
&= N\int_{t^c}^{b} h'(\tilde{\theta})\frac{\partial\tilde{\theta}}{\partial t^c}f(t)dt. \qquad (25)
\end{aligned}$$

Note that in the linear cost case in which $h' = 1$, we have shown that

$$\int_{t^c}^{b} \frac{\partial\tilde{\theta}}{\partial t^c}f(t)dt > 0. \qquad (26)$$

The above inequality implies that there exists some $t^* \in (t^c, b)$ such that $\frac{\partial\tilde{\theta}}{\partial t^c}|_{t=t^*} > 0$. If we can show that $\phi(t,t^c) := \frac{\partial\tilde{\theta}(t,t^c)}{\partial t^c}$ satisfies $\phi(t^c,t^c) \leq 0$, and $\phi(t,t^c)$, as a function of t, is either increasing or first decreasing and then increasing, then it implies that there exists a unique $t^{**} \in [t^c, b)$ such

that $\phi(t,t^c) \leq 0$ when $t \leq t^{**}$ and $\phi(t,t^c) > 0$ when $t > t^{**}$. Then since $h'' \geq 0$, using similar argument as that in the proof of the cross-scenario transfer to prove (23) from (24), (25) follows from (26).

We first show that $\phi(t^c,t^c) \leq 0$. Note that when $n \geq 2$, $\gamma_n(t^c,t^c) = 0$; $\frac{\partial \gamma_2(t,t^c)}{\partial t^c}|_{t=t^c} = -\frac{t^c f(t^c)}{1-F(t^c)}$; and $\frac{\partial \gamma_k(t,t^c)}{\partial t^c}|_{t=t^c} = 0$ when $k \geq 3$. Thus,[40]

$$\begin{aligned}
\phi(t^c,t^c) &= \frac{\partial \tilde{\theta}(t,t^c)}{\partial t^c}|_{t=t^c} = \sum_{k=n}^{N}[\frac{dp_k(t^c)}{dt^c}\gamma_k(t,t^c) + p_k(t^c)\frac{\partial \gamma_k(t,t^c)}{\partial t^c}]|_{t=t^c} \\
&\quad -[n\frac{d}{dt^c}(c - \sum_{k=1}^{n-1}\frac{p_k(t^c)}{k})\gamma_n(t,t^c) + n(c - \sum_{k=1}^{n-1}\frac{p_k(t^c)}{k})\frac{\partial \gamma_n(t,t^c)}{\partial t^c}]|_{t=t^c} \\
&= p_n(t^c)\frac{\partial \gamma_n(t,t^c)}{\partial t^c}|_{t=t^c} - n(c - \sum_{k=1}^{n-1}\frac{p_k(t^c)}{k})\frac{\partial \gamma_n(t,t^c)}{\partial t^c}|_{t=t^c}.
\end{aligned}$$

By the definition of t_n, $c - \sum_{k=1}^{n-1}\frac{p_k(t^c)}{k} \leq \frac{p_n(t^c)}{n}$. Therefore, if $n = 2$, then $\phi(t^c,t^c) = \frac{t^c f(t^c)}{1-F(t^c)}[2(c - p_1(t^c)) - p_2(t^c)] \leq \frac{t^c f(t^c)}{1-F(t^c)}[p_2(t^c) - p_2(t^c)] = 0$; if $n > 2$, then $\phi(t^c,t^c) = 0$ since $\frac{\partial \gamma_n(t,t^c)}{\partial t^c}|_{t=t^c} = 0$ when $n > 2$.

The next step is to show that $\phi(t,t^c)$, as a function of t, is either increasing or first decreasing and then increasing. To this end, we shall examine the sign of $\frac{\partial \phi}{\partial t} = \frac{\partial^2 \tilde{\theta}(t,t^c)}{\partial t \partial t^c}$. Notice that

$$\begin{aligned}
\frac{\partial \tilde{\theta}(t,t^c)}{\partial t} &= \sum_{k=n}^{N}(k-1)p_k(t^c)tG^{k-2}(t,t^c)g(t,t^c) \\
&\quad -n(n-1)(c - \sum_{k=1}^{n-1}\frac{p_k(t^c)}{k})tG^{n-2}(t,t^c)g(t,t^c) \\
&= (N-1)tf(t)\sum_{k=n}^{N}\binom{N-2}{k-2}(F(t)-F(t^c))^{k-2}F^{N-k}(t^c) \\
&\quad -n(n-1)(c - \sum_{k=1}^{n-1}\frac{p_k(t^c)}{k})tG^{n-2}(t,t^c)g(t,t^c).
\end{aligned}$$

[40] Recall that $K_n^*(t^c)p_n(t^c) = c - \sum_{k=1}^{n-1}\frac{p_k(t^c)}{k}$.

Thus,[41]

$$\frac{\partial^2\tilde{\theta}(t,t^c)}{\partial t\partial t^c} = \frac{\partial^2\tilde{\theta}(t,t^c)}{\partial t^c\partial t} = -(N-1)(N-2)tf(t)f(t^c)\binom{N-3}{n-3}(F(t)-F(t^c))^{n-3}F^{N-n}(t^c)$$

$$-n(n-1)\frac{\partial}{\partial t^c}[(c-\sum_{k=1}^{n-1}\frac{p_k(t^c)}{k})tG^{n-2}(t,t^c)g(t,t^c)].$$

In the proof of the linear case in Lemma 7, we have shown that[42]

$$\frac{d}{dt^c}\sum_{k=1}^{n-1}\frac{p_k(t^c)}{k} = \frac{f(t^c)}{1-F(t^c)}(p_n(t^c)+\sum_{k=2}^{n-1}\frac{p_k(t^c)}{k}).$$

Now note that

$$\frac{\partial}{\partial t^c}[G^{n-2}(t,t^c)g(t,t^c)]$$

$$= \left[\begin{array}{c} -(n-2)G^{n-3}(t,t^c)\frac{f(t^c)(1-G(t,t^c))}{1-F(t^c)}g(t,t^c) \\ +G^{n-2}(t,t^c)\frac{f(t)f(t^c)}{(1-F(t^c))^2} \end{array}\right]$$

$$= \frac{f(t^c)g(t,t^c)}{1-F(t^c)}[-(n-2)G^{n-3}(t,t^c)(1-G(t,t^c))+G^{n-2}(t,t^c)].$$

Thus,

$$\frac{\partial}{\partial t^c}[(c-\sum_{k=1}^{n-1}\frac{p_k(t^c)}{k})tG^{n-2}(t,t^c)g(t,t^c)]$$

$$= -\frac{f(t^c)}{1-F(t^c)}(p_n(t^c)+\sum_{k=2}^{n-1}\frac{p_k(t^c)}{k})tG^{n-2}(t,t^c)g(t,t^c)$$

$$+(c-\sum_{k=1}^{n-1}\frac{p_k(t^c)}{k})t\frac{f(t^c)g(t,t^c)}{1-F(t^c)}[-(n-2)G^{n-3}(t,t^c)(1-G(t,t^c))+G^{n-2}(t,t^c)]$$

$$= \frac{tf(t^c)g(t,t^c)G^{n-3}(t,t^c)}{1-F(t^c)}\left\{\begin{array}{c}[(n-1)(c-\sum_{k=1}^{n-1}\frac{p_k(t^c)}{k})-p_n(t^c)-\sum_{k=2}^{n-1}\frac{p_k(t^c)}{k}]G(t,t^c) \\ -(n-2)(c-\sum_{k=1}^{n-1}\frac{p_k(t^c)}{k})\end{array}\right\}.$$

[41] Define $\binom{m}{j}=0$ if $j<0$ or $m\leq 0$.

[42] Define $\sum_{j=m_1}^{m_2}\phi(j)=0$ if $m_2<m_1$ for any function ϕ.

So

$$\begin{aligned}
\frac{\partial \phi}{\partial t} &= -(N-1)(N-2)tf(t)f(t^c)\binom{N-3}{n-3}(F(t)-F(t^c))^{n-3}F^{N-n}(t^c) \\
&\quad -\frac{n(n-1)tf(t^c)g(t,t^c)G^{n-3}(t,t^c)}{1-F(t^c)}\cdot\{[(n-1)(c-\sum_{k=1}^{n-1}\frac{p_k(t^c)}{k}) \\
&\quad -p_n(t^c)-\sum_{k=2}^{n-1}\frac{p_k(t^c)}{k}]G(t,t^c)-(n-2)(c-\sum_{k=1}^{n-1}\frac{p_k(t^c)}{k})\} \\
&= \frac{n(n-1)tf(t^c)g(t,t^c)G^{n-3}(t,t^c)}{1-F(t^c)}\{-\frac{n-2}{n}p_n(t^c)+[p_n(t^c)+\sum_{k=2}^{n-1}\frac{p_k(t^c)}{k} \\
&\quad -(n-1)(c-\sum_{k=1}^{n-1}\frac{p_k(t^c)}{k})]G(t,t^c)+(n-2)(c-\sum_{k=1}^{n-1}\frac{p_k(t^c)}{k})\}.
\end{aligned}$$

Then when $n=2$,

$$\frac{\partial \phi}{\partial t}=\frac{2tf(t^c)g(t,t^c)}{1-F(t^c)}(p_1(t^c)+p_2(t^c)-c).$$

By the definition of t_2, we have $c \le p_1(t^c)+\frac{p_2(t^c)}{2}$. Therefore,

$$\frac{\partial \phi}{\partial t} \ge \frac{tf(t^c)g(t,t^c)p_2(t^c)}{1-F(t^c)} \ge 0,$$

which means $\phi(t,t^c)$ is increasing in t.

Define $\lambda(t^c) := p_n(t^c)+\sum_{k=2}^{n-1}\frac{p_k(t^c)}{k}-(n-1)(c-\sum_{k=1}^{n-1}\frac{p_k(t^c)}{k})$, $\mu(t^c) := \frac{n-2}{n}[n(c-\sum_{k=1}^{n-1}\frac{p_k(t^c)}{k})-p_n(t^c)]$. When $n>2$ and $t>t^c$, $\frac{\partial \phi}{\partial t}$ has the same sign as

$$\eta(t) := \lambda(t^c)G(t,t^c)+\mu(t^c).$$

By the definition of t_n,

$$0 \le c-\sum_{k=1}^{n-1}\frac{p_k(t^c)}{k} \le \frac{p_n(t^c)}{n}.$$

Note that $\eta(t)$ is linear in $G(t,t^c)$. The "slope"

$$\lambda(t^c) \ge p_n(t^c)+\sum_{k=2}^{n-1}\frac{p_k(t^c)}{k}-(n-1)\cdot\frac{p_n(t^c)}{n}=\sum_{k=1}^{n}\frac{p_k(t^c)}{k} \ge c > 0.$$

And the "intercept" $\mu(t^c)=0$ when $n=2$; when $n>2$, since

$$0 \le n(c-\sum_{k=1}^{n-1}\frac{p_k(t^c)}{k}) \le p_n(t^c),$$

we have

$$-\frac{(n-2)p_n(t^c)}{n} \leq \mu(t^c) \leq 0.$$

Also note that

$$\begin{aligned}
\eta(b) &= \lambda(t^c) + \mu(t^c) = p_n(t^c) + \sum_{k=2}^{n-1} \frac{p_k(t^c)}{k} - (n-1)(c - \sum_{k=1}^{n-1} \frac{p_k(t^c)}{k}) \\
&\quad +(n-2)(c - \sum_{k=1}^{n-1} \frac{p_k(t^c)}{k}) - \frac{(n-2)p_n(t^c)}{n} \\
&= \frac{2}{n} p_n(t^c) + \sum_{k=2}^{n-1} \frac{p_k(t^c)}{k} - (c - \sum_{k=1}^{n-1} \frac{p_k(t^c)}{k}) \\
&\geq \frac{2}{n} p_n(t^c) + \sum_{k=2}^{n-1} \frac{p_k(t^c)}{k} - \frac{p_n(t^c)}{n} \\
&= \sum_{k=2}^{n} \frac{p_k(t^c)}{k} > 0.
\end{aligned}$$

Therefore, there exists a unique $\tilde{t}^* \in [t^c, b)$ such that $\eta(t) \leq 0$ when $t \leq \tilde{t}^*$; $\eta(t) > 0$ when $t > \tilde{t}^*$. This completes the proof. □

6.3 The Convex Cost Case

In this subsection, we provide an example that winner-take-all is optimal when there is no cost of entry, while it is not when the entry cost $c > 0$. Assume that $F(t) = t - 1, t \in [1, 2]$, i.e., the contestants' types follow the uniform distribution with support $[1, 2]$. Suppose that the contestant's cost of exerting effort e when his type is t is $\tilde{h}(e)/t$, where $\tilde{h}(0) = 0$, $\tilde{h}' > 0$, and $\tilde{h}'' \geq 0$. Let $h(\cdot) = h^{-1}(\cdot)$, then $h' > 0$ and $h'' \leq 0$. Assume that $N = 3$, $h(t) = t^{0.8}$, and entry cost $c = 0.25$.

When there is no entry cost, similarly to Moldovanu and Sela [9], an equivalent condition for winner-take-all to be optimal can be characterized. To this end, assume that the first prize is $1 - v$ and the second prize is v, where $v \in [0, 0.5]$.[43] Then similar to the case with concave cost, the

[43] It is obvious that there is no loss of generality to assume that the third prize is zero.

bidding function is simply a concave transformation of the linear cost case:

$$\begin{aligned} e_0(t,v) &= h\{(1-v)(tF^2(t)-\int_1^t F^2(s)ds)+v[2tF(t)(1-F(t))-\int_1^t 2F(s)(1-F(s))ds]\} \\ &= h\{\underbrace{v[t(t-1)(5-3t)-\int_1^t (s-1)(5-3s)ds]+t(t-1)^2-\int_1^t (s-1)^2 ds}_{e_L(t,v)}\}. \end{aligned}$$

Note that $e_L(t,v)$ is the bidding function in the linear cost case which is linear in v. Now the expected total effort with free entry is

$$TE_0(v) = 3\int_1^2 e_0(t,v)dt = 3\int_1^2 h(e_L(t,v))dt.$$

Take derivative with respect to v:

$$TE_0'(v) = 3\int_1^2 h'(e_L(t,v))e_L'(t,v)dt.$$

Take derivative with respect to v again

$$TE_0''(v) = 3\int_1^2 h''(e_L(t,v))e_L'(t,v)dt.$$

Since the bidding function in the linear cost case $e_L(t,v)$ is weakly increasing and since $h'' \leq 0$, $TE_0''(v) \leq 0$. As a result, winner-take-all is optimal if and only if $TE_0'(0) \leq 0$. That is,

$$\int_1^2 h'[t(t-1)^2-\int_1^t (s-1)^2 ds][t(t-1)(5-3t)-\int_1^t (s-1)(5-3s)ds]dt \leq 0.$$

When $h(t) = t^{0.8}$, the left hand side is (approximately) equal to -0.0431679, which is negative. Therefore, winner-take-all is optimal when there is no entry cost.

Now we come to the situation where each contestant needs to incur an entry cost of 0.25 to participate. Like what we did before, assume that the minimum prize vector is $\mathbf{K} = (K_1, K_2, K_3)$, where $K_n \in [0, \frac{1}{n}]$, $n = 1, 2, 3$. Again, the threshold type always bids zero so that the minimum prize vector that implements the entry threshold $t^c \in [1, 2]$ satisfies

$$K_3(1-F(t^c))^2 + 2K_2F(t^c)(1-F(t^c)) + K_3F^2(t^c) = 0.25.$$

Assume that the prize allocation vector $\mathbf{W} = (\mathbf{W}_1, \mathbf{W}_2, \mathbf{W}_3)$, where $\mathbf{W}_1 = K_1$, $\mathbf{W}_2 = (1 - K_2, K_2)$, and $\mathbf{W}_3 = (1 - v - K_3, v, K_3)$, where v is the second prize in scenario 3. Similar to the concave cost case, the bidding function can be expressed as

$$e(t, \mathbf{W}, t^c) = h[\sum_{n=1}^{3} p_n(t^c)e^{(n)}(t, \mathbf{W}_n, t^c)],$$

where $p_n(t^c) = \binom{2}{n-1}(1-F(t^c))^{n-1}F^{3-n}(t^c)$ is the probability of scenario n happening from an entrant's perspective and $e^{(n)}(t, \mathbf{W}_n, t^c)$ is given in Lemma 5. Then the expected total effort can be written as

$$\begin{aligned} TE(\mathbf{W}, t^c) &= \sum_{n=1}^{3} \binom{3}{n} (1-F(t^c))^n F^{n-3}(t^c) n \int_{t^c}^{2} h[\sum_{n=1}^{3} p_n(t^c) e^{(n)}(t, \mathbf{W}_n, t^c)] g(t, t^c) dt \\ &= 3 \sum_{n=1}^{3} (1-F(t^c)) \binom{2}{n-1} \int_{t^c}^{2} h[\sum_{n=1}^{3} p_n(t^c) e^{(n)}(t, \mathbf{W}_n, t^c)] g(t, t^c) dt \\ &= 3 \int_{t^c}^{2} h[\sum_{n=1}^{3} p_n(t^c) e^{(n)}(t, \mathbf{W}_n, t^c)] f(t) dt. \end{aligned}$$

Winner-take-all corresponds to the prize allocation vector $\mathbf{W}_1 = 1, \mathbf{W}_2 = (1, 0), \mathbf{W}_3 = (1, 0, 0)$, and the entry threshold $t^c = 1.5$. The expected total effort under winner-take-all is (approximately) 0.905178. Now consider the entry threshold $t^c = 1.49$ and the prize vector $\mathbf{W}_1 = 1, \mathbf{W}_2 = (0.9801921, 0.0198079), \mathbf{W}_3 = (1, 0, 0)$. The expected total effort is (approximately) 0.905821, which is higher than that under winner-take-all. □

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